Group classification of nonlinear wave equations

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Abstract

We perform complete group classification of the general class of quasi linear wave equations in two variables. This class may be seen as a broad generalization of the nonlinear d'Alembert, Liouville, sin/sinh-Gordon and Tzitzeica equations. In this way we derived a number of new genuinely nonlinear invariant models with high symmetry properties. In particular, we obtain four classes of nonlinear wave equations admitting five-dimensional invariance groups. Applying the symmetry reduction technique we construct multi-parameter families of exact solutions of those wave equations.

Introduction

More than century ago Sophus Lie introduced the concept of continuous transformation group into mathematical physics and mechanics. His initial motivation was to develop a theory of integration of ordinary differential equations enabling to answer the basic questions, like, why some equations are integrable and others are not. His fundamental results obtained on this way, can be seen as a far reaching generalization of the Galua's and Abel's theory of solubility of algebraic equations by radicals. Since that time the Lie's theory of continuous transformation groups has become applicable to an astonishingly wide range of mathematical and physical problems.

It was Lie who was the first to utilize group properties of differential equations for constructing of their exact solutions. In particular, he computed the maximal invariance group of the one-dimensional heat conductivity equation and applied this symmetry to construct its explicit solutions. Saying it the modern way, he performed symmetry reduction of the heat equation. Since late 1970s symmetry reduction becomes one of the most popular tools for solving nonlinear partial differential equations (PDEs).

By now symmetry properties of the majority of fundamental equations of mathematical and theoretical physics are well known. It turns out that for the most part these equations admit wide symmetry groups. Especially this is the case for linear PDEs and it is this rich symmetry that enables developing a variety of efficient methods for mathematical analysis of linear differential equations. However, linear equations give mathematical description of physical, chemical or biological processes in a first approximation only. To provide a more detailed and precise description a mathematical model has to incorporate nonlinear terms. Note that some important equations (for example, the Yang-Mills equations) of theoretical physics are essentially nonlinear in a sense that they have no linearized version.

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Hyperbolic type second-order nonlinear PDEs in two independent variables play a fundamental role in modern mathematical physics. Equations of this type are utilized to describe different types of wave propagation. They are used in differential geometry, in various fields of hydro- and gas dynamics, chemical technology, super conductivity, crystal dislocation to mention only a few applications areas. Surprisingly the list of equations utilized is rather narrow. In fact, it is comprised by the Liouville, sine/sinh-Gordon, Goursat, d'Alembert and Tzitzeica equations and a couple of others. Popularity of these very models has a natural group-theoretical interpretation, namely, all of them have non-trivial Lie or Lie-Bäcklund symmetry. By his very reason some of them are integrable by the inverse problem methods (see, e.g.,[1]–[3]) or are linearizable [4]–[6]) and completely integrable [7, 8].

In this connection it seems a very important problem to select from the reasonably extensive class of nonlinear hyperbolic type PDEs those enjoying the best symmetry properties. Saying 'reasonably extensive" we mean that this class should contain the above enumerated equations as particular cases on the one hand, and on the other it should contain a wide variety of new invariant models of potential interest for applications. The list of the so obtained invariant equations will contain candidates for realistic nonlinear mathematical models of the physical and chemical processes enumerated above.

The history of group classification methods goes back to Lie itself. Probably, the very first paper on this subject is [9], where Lie proves that a linear two-dimensional second-order PDE may admit at most a three-parameter invariance group (apart from the trivial infinite-parameter symmetry group, which is due to linearity).

The modern formulation of the problem of group classification of PDEs was suggested by Ovsyannikov in [10]. He developed a regular method (we will refer to it as to Lie-Ovsyannikov method) for classifying differential equations with non-trivial symmetry and performed complete group classification of the nonlinear heat conductivity equation. In a number of subsequent publications more general types of nonlinear heat equations were classified (review of these results can be found in [11]).

However, even a very quick analysis of the papers on group classification of PDEs reveals that the overwhelming majority of them deal with equations whose arbitrary elements (functions) depend on one variable only. There is a deep reason for this fact, which is that Lie-Ovsyannikov method becomes inefficient for PDEs containing arbitrary functions of several variables. To achieve a complete classification one either needs to specify the transformation group realization or somehow restrict arbitrariness of functions contained in the equation under study.

Recently, we developed an efficient approach enabling to overcome this difficulty for low dimensional PDEs [12, 13]. Utilizing it approach we have obtained a final solution of the problem of group classification of the general quasi-linear heat conductivity equations in two independent variables.

In this paper we apply the approach mentioned above to perform group classification of the most general quasi-linear hyperbolic type PDE in two independent variables.

1 Group classification algorithm

We begin this section by formulating the problem to be solved. Then we present a brief review of the already known results. Finally we give a short description of our approach to group classification of PDEs (for the detailed account of the necessary facts, see ([13])).

While classifying a given class of differential equations into subclasses, one can use different classifying features, like linearity, order, the number of independent or dependent variables, etc. In group analysis of differential equations the principal classifying features are symmetry properties of equations under study. This means that classification objects are equations together with their symmetry groups. This point of view is based on the well-known fact that any PDE admits a (possibly trivial) Lie transformation group. And what is more, any transformation group corresponds to a class of PDEs, which are invariant under this group. So to perform group classification of a class of PDEs means describing all possible (inequivalent) pairs (PDE, maximal invariance group), where PDE should belong to the class of equations under consideration.

We perform group classification of the following class of quasi-linear wave equations

$$u_{tt} = u_{xx} + F(t, x, u, u_x). \tag{1.1}$$

Here F is an arbitrary smooth function, $u = u(t, x)$. Hereafter we adopt notations $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}, \ldots$.

Our aim is describing *all* equations of the form (1.1) that admit nontrivial symmetry groups. The challenge of this task is in the word *all*. If, for example, we fix the desired invariance group, then the classification problem becomes a University textbook exercise on Lie group analysis. A slightly more cumbersome (but still tractable with the standard Lie-Ovsyannikov approach) is the problem of group classification of equation with arbitrary functions of, at most, one variable.

As equations invariant under similar Lie groups are identical within the group-theoretic framework, it makes sense to consider non similar transformation groups [14, 15] only. The important example of similar Lie groups is provided by Lie transformation groups obtained one from another by a suitable change of variables. Consequently, equations obtained one from another by a change of variables have similar symmetry groups and cannot be distinguished within the group-theoretical viewpoint. That is why, we perform group classification of (1.1) within a change of variables preserving the class of PDEs (1.1).

The problem of group classification of linear hyperbolic type equation

$$u_{tx} + A(t, x)u_t + B(t, x)u_x + C(t, x)u = 0 \tag{1.2}$$

with $u = u(t, x)$, was solved by Lie [9] (see, also, [16]). In view of this fact, we consider only those equations of the form (1.1) which are not (locally) equivalent to the linear equation (1.2).

As we have already mentioned in Introduction, the Lie-Ovsyannikov method of group classification of differential equations has been suggested in [10]. Utilizing this method enabled solving group classification problem for a number of important one-dimensional nonlinear wave equations:

$$\begin{aligned}
u_{tt} &= u_{xx} + F(u); && [17]-[19]\\
u_{tt} &= [f(u)u_x]_x; && [20]-[22]\\
u_{tt} &= f(u_x)u_{xx}; && [22,\ 23]\\
u_{tt} &= F(u_x)u_{xx} + H(u_x); && [24]\\
u_{tt} &= F(u_{xx}); && [22]\\
u_{tt} &= u_x^m u_{xx} + f(u); && [25]\\
u_{tt} + f(u)u_t &= (g(u)u_x)_x + h(u)u_x. && [26]
\end{aligned}$$

Analysis of the above list shows immediately that arbitrary elements (= arbitrary functions) depend on one variable, at most. This is not coincidental, indeed, the Lie-Ovsyannikov approach works smoothly for the case when the arbitrary elements are functions of one variable. The reason for this is that the obtained system of determining equations is still over-determined and can be effectively solved within the same ideology used while computing maximal symmetry group of PDEs containing no arbitrary elements.

The situation becomes much more complicated for the case when arbitrary elements are functions of two (or more) arguments. By this very reason group classification of nonlinear wave equations

$$\begin{aligned}
u_{tt} &= (f(x, u)u_x)_x; && [27]\\
u_{tt} + \lambda u_{xx} &= g(u, u_x); && [28,\ 29]\\
u_{tt} &= [f(u)u_x + g(x, u)]_x; && [30]-[32]\\
u_{tt} &= f(x, u_x)u_{xx} + g(x, u_x) && [33]
\end{aligned}$$

is not complete.

We suggest new efficient approach to the problem of group classification of low dimensional PDEs in [12, 13]. This approach is based on Lie-Ovsyannikov infinitesimal method and classification results for abstract finite-dimensional Lie algebras. It enables us to obtain the complete solution of group classification problem for the general heat equation with a nonlinear source

$$u_t = u_{xx} + F(t, x, u, u_x).$$

Later on, we perform complete group classification of the most general quasi-linear evolution equation [34]–[36]

$$u_t = f(t, x, u, u_x)u_{xx} + g(t, x, u, u_x).$$

We use the above approach to achieve a complete solution of the group classification problem for the class of equations (1.1).

Our approach to group classification of the class of PDEs (1.1) consists of the following steps (for more details, see [36]):

I. Using the infinitesimal Lie method we derive the system of determining equations for coefficients of the first-order operator that generates symmetry group of equation (1.1) (Note that the determining equations which explicitly depend on the function F and its derivatives are called classifying equations). Integrating equations that do not depend on F we obtain the form of the most general infinitesimal operator admitted by equation (1.1) under arbitrary F. Another task of this step is calculating the equivalence group $\mathcal{E}$ of the class of PDEs (1.1).

II. We construct all realizations of Lie algebras A_n of the dimension $n \leq 3$ in the class of operators obtained at the first step within the equivalence relation defined by transformations from the equivalence group $\mathcal{E}$. Inserting the so obtained operators into classifying equations we select those realizations that can be symmetry algebras of a differential equation of the form (1.1).

III. We perform extension of the realizations constructed at the previous step to realizations of higher dimensional ($n > 3$) Lie algebras. Since extending symmetry algebras results in reducing arbitrariness of the function F, at some point this function will contain either arbitrary functions of at most one variable or arbitrary constants. At this point, we apply the standard classification method, which is due to Lie and Ovsyannikov, to derive the maximal symmetry group of the equation under study thus completing its group classification.

As a result of performing the above enumerated steps we get the complete list of inequivalent equations of the form (1.1) together with their maximal (in Lie's sense) symmetry algebras.

We say that the group classification problem is completely solved when it is proved that

1) The constructed symmetry algebras are maximal invariance algebras of the equations under consideration;

2) The list of invariant equations contains only inequivalent ones, namely, no equation can be transformed into another one from the list by a transformation from the equivalence group $\mathcal{E}$.

2 Preliminary group classification of equation (1.1)

According to the above algorithm we are looking for infinitesimal operator of symmetry group of equation (1.1) in the form

$$Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \tag{2.1}$$

where τ, ξ, η are smooth functions defined on an open domain Ω of the space $V = \mathbb{R}^2 \times \mathbb{R}^1$ of independent $\mathbb{R}^2 = \langle t, x \rangle$ and dependent $\mathbb{R}^1 = \langle u \rangle = u(t,x)$ variables.

Operator (2.1) generates one-parameter invariance group of (1.1) iff its coefficients $\tau, \xi, \eta, \epsilon$ satisfy the equation (Lie's invariance criterion)

$$\varphi^{tt} - \varphi^{xx} - \tau F_t - \xi F_x - \eta F_u - \varphi^x F_{u_x} \Big|_{(1.1)} = 0, \tag{2.2}$$

where

$$\begin{aligned}
\varphi^t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\
\varphi^x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\
\varphi^{tt} &= D_t(\varphi^t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\
\varphi^{xx} &= D_x(\varphi^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi)
\end{aligned}$$

and D_t, D_x are operators of total differentiation with respect to the variables t, x. As customary, by writing $\Big|_{(1.1)}$ we mean that one needs to replace u_{tt} and its differential consequences with the expression $u_{xx} + F$ and its differential consequences in (2.2).

After a simple algebra we represent (2.2) in the form of system of four PDEs:

$$\begin{array}{ll}
(1) & \xi_u = \tau_u = \eta_{uu} = 0, \\
(2) & \tau_t - \xi_x = 0, \quad \xi_t - \tau_x = 0, \\
(3) & 2\eta_{tu} + \tau_x F_{u_x} = 0, \\
(4) & \eta_{tt} - \eta_{xx} - 2u_x \eta_{xu} + [\eta_u - 2\tau_t]F - \tau F_t - \xi F_x \\
 & -\eta F_u - [\eta_x + u_x(\eta_u - \xi_x)]F_{u_x} = 0.
\end{array} \tag{2.3}$$

It follows immediately from (1) that $\tau = \tau(t,x),\ \xi = \xi(t,x), \eta = h(t,x)u + r(t,x)$. In the sequel we differentiate between the cases $F_{u_x u_x} \neq 0$ and $F_{u_x u_x} = 0$.

Case $F_{u_x u_x} \neq 0$. It follows from (3) that $\tau_x = h_t = 0$. Taking into account this fact and equation (2) we obtain $\tau = \lambda t + \lambda_1, \xi = \lambda x + \lambda_2,\ h = h(x)$, where $\lambda, \lambda_1, \lambda_2$ are arbitrary real constants.

Case $F_{u_x u_x} = 0$. If this is the case, then $F = g(t,x,u)u_x + f(t,x,u)$, where f and g are arbitrary smooth functions.

Given the condition $g_u \neq 0$, it follows from (3) that $\tau_x = h_t = 0$. So that taking into account equation (2) we arrive at the expressions τ, ξ, h given in the previous case.

If $g_u = 0$, then $f_{uu} \neq 0$ (otherwise equation (1.1) becomes linear).

Let $g \equiv 0$. It follows from (3) that $\eta = h(x)u + r(t,x)$. Equation (4) now reads as

$$r_{tt} - r_{xx} - 2h' u_x + [h - 2\tau_t]f - \tau f_t - \xi f_x - [hu + r]f_u = 0.$$

As functions τ, ξ, h, r, f do not depend on u_x, $h' = 0$. Hence $\eta = mu + r(t,x)$, where m is an arbitrary real constant.

Furthermore, if $g = g(t,x) \neq 0$, then it is straightforward to verify that system of equations ((3), (4)) is equivalent to the following equations:

$$\begin{aligned}
&2h_t = -\tau_x g, \quad 2h_x = -\tau_t g - \tau g_t - \xi g_x, \\
&(h_{tt} - h_{xx})u + r_{tt} - r_{xx} + f[h - 2\tau_t] \quad - \tau f_t - \xi f_x - [hu + r]f_u - (h_x u + r_x)g = 0.
\end{aligned}$$

Integrating (2) yields $\tau = \varphi(\theta) + \psi(\sigma)$, $\xi = -\varphi(\theta) + \psi(\sigma)$, where φ, ψ are arbitrary smooth functions of $\theta = t - x, \sigma = t + x$ and we arrive at the following theorem.

Theorem 1 *Provided $F_{u_x u_x} \neq 0$, the maximal invariance group of equation (1.1) is generated by the following infinitesimal operator:*

$$Q = (\lambda t + \lambda_1)\partial_t + (\lambda x + \lambda_2)\partial_x + [h(x)u + r(t,x)]\partial_u, \tag{2.4}$$

where $\lambda, \lambda_1, \lambda_2$ are real constants and $h = h(x),\ r = r(t,x),\ F = F(t,x,u,u_x)$ are functions obeying the constraint

$$\begin{aligned} & r_{tt} - r_{xx} - \frac{d^2h}{dx^2}u - 2\frac{dh}{dx}u_x + (h - 2\lambda)\,F \\ & -(\lambda t + \lambda_1)\,F_t - (\lambda x + \lambda_2)\,F_x - (hu + r)\,F_u \\ & -(r_x + \frac{dh}{dx}u + (h-\lambda)u_x)\,F_{u_x} = 0. \end{aligned} \tag{2.5}$$

If $F = g(t,x,u)u_x + f(t,x,u),\ g_u \neq 0$, then the maximal invariance group of equation (1.1) is generated by infinitesimal operator (2.4), where $\lambda, \lambda_1, \lambda_2$ are real constants h, r, g, f are functions satisfying system of two equations

$$\begin{aligned} & -2h' - \lambda g = (\lambda t + \lambda_1)g_t + (\lambda x + \lambda_2)g_x + (hu + r)g_u, \\ & -h''u + r_{tt} - r_{xx} + (h - 2\lambda)f = (\lambda t + \lambda_1)f_t + (\lambda x + \lambda_2)f_x \\ & \qquad +(hu + r)f_u + g(h'u + r_x). \end{aligned} \tag{2.6}$$

Next, if $F = g(t,x)u_x + f(t,x,u),\ q \not\equiv 0,\ f_{uu} \neq 0$, then the infinitesimal operator of the invariance group of equation (1.1) reads as

$$Q = \tau(t,x)\partial_t + \xi(t,x)\partial_x + (h(t,x)u + r(t,x))\partial_u,$$

where τ, ξ, h, r, g, f are functions satisfying system of PDEs

$$\begin{aligned} & \tau_t - \xi_x = 0, \quad \xi_t - \tau_x = 0, \\ & 2h_t = -\tau_x g, \quad 2h_x = -\tau_t g - \tau g_t - \xi g_x, \\ & (h_{tt} - h_{xx})u + r_{tt} - r_{xx} + f(h - 2\tau_t) - \tau f_t \\ & -\xi f_x - (hu + r)f_u - (h_x u + r_x)g = 0. \end{aligned}$$

Finally, if $F = f(t,x,u),\ f_{uu} \neq 0$, then the maximal invariance group of equation (1.1) is generated by infinitesimal operator

$$Q = [\varphi(\theta) + \psi(\sigma)]\partial_t - [\varphi(\theta) - \psi(\sigma)]\partial_x + [ku + r(t,x)]\partial_u,$$

where $k \in \mathbb{R},\ \theta = t - x,\ \sigma = t + x$ and functions φ, ψ, r, f and constant k satisfy the following equation:

$$\begin{aligned} & r_{tt} - r_{xx} + [k - 2\varphi' - 2\psi']f - (\varphi + \psi)f_t + \\ & +(\varphi - \psi)f_x - (ku + r)f_u = 0, \quad \varphi' = \frac{d\varphi}{d\theta}, \quad \psi' = \frac{d\psi}{d\theta}. \end{aligned}$$

Summing up the above considerations we conclude that the problem of group classification of equation (1.1) reduces to the one of classifying equations of more specific forms

$$u_{tt} = u_{xx} + F(t,x,u,u_x), \quad F_{u_x u_x} \neq 0; \tag{2.7}$$

$$u_{tt} = u_{xx} + g(t,x,u)u_x + f(t,x,u), \quad g_u \neq 0; \tag{2.8}$$

$$u_{tt} = u_{xx} + g(t,x)u_x + f(t,x,u), \quad g \neq 0,\ f_{uu} \neq 0; \tag{2.9}$$

$$u_{tt} = u_{xx} + f(t,x,u), \quad f_{uu} \neq 0. \tag{2.10}$$

Consider the last two equations. By the change of variables

$$\bar{t} = t - x, \quad \bar{x} = t + x, \quad u = v(\bar{t}, \bar{x})$$

we reduce them to equations

$$\begin{aligned} v_{\bar{t}\bar{x}} &= \frac{1}{4} f(\bar{t}, \bar{x}, v), \\ v_{\bar{t}\bar{x}} &= -\frac{1}{4} g(\bar{t}, \bar{x})(v_{\bar{t}} - v_{\bar{x}}) + \frac{1}{4} f(\bar{t}, \bar{x}, v). \end{aligned} \tag{2.11}$$

Now, making the change of variables

$$\tilde{t} = \bar{t}, \quad \tilde{x} = \bar{x}, \quad \tilde{v}(\tilde{t}, \tilde{x}) = \Lambda(\bar{t}, \bar{x}) v,$$

where $\Lambda = \exp\left[-\frac{1}{4}\int g(\bar{t}, \bar{x}) d\bar{x}\right]$, we transform (2.11) to become

$$\tilde{v}_{\tilde{t}\tilde{x}} = \left(\frac{1}{4} g - \Lambda^{-1}\Lambda_{\bar{t}}\right) \tilde{v}_{\tilde{x}} - \frac{1}{4} g \Lambda^{-1}\Lambda_{\bar{t}} \tilde{v} + \frac{1}{4}\Lambda^{-1}\Lambda_{\bar{x}} g \tilde{v} + \Lambda^{-1} f.$$

Hence we conclude that the following assertion holds true.

Assertion 1 *The problem of group classification of equations (2.9), (2.10) is equivalent to the one of classifying equations*

$$u_{tx} = g(t, x) u_x + f(t, x, u), \ g_x \neq 0, \ f_{uu} \neq 0; \tag{2.12}$$

$$u_{tx} = f(t, x, u), \quad f_{uu} \neq 0. \tag{2.13}$$

Note that condition $g_x \neq 0$ is essential, since otherwise (2.12)is locally equivalent (2.13).

Summing up, we conclude that the problem of group classification of (1.1) reduces to classifying more particular classes of PDEs (2.7), (2.8, (2.12), (2.13). In what follows, we provide full calculation details for equations (2.9) and (2.10) only. The reason is just to save space and still be able to present all details of the algorithm.

First, we consider equations (2.8), (2.12), (2.13).

3 Group classification of equation (2.8)

According to Theorem 1 invariance group of equation (2.8) is generated by infinitesimal operator (2.4). And what is more, the real constants $\lambda, \lambda_1, \lambda_2$ and functions h, r, g, f satisfy equations (2.6). System (2.6) is to be used to specify both the form of functions f, g from (2.8) and functions h, r and constants $\lambda, \lambda_1, \lambda_2$ in (2.4). It is called the determining (sometimes classifying) equations.

Efficiency of the Lie method for calculation of maximal invariance group of PDE is essentially based on the fact that routinely this system is over-determined. This is clearly not the case, since we have only one equation for four (!) arbitrary functions and three of the latter depend on two variables. By this very reason the direct application of Lie approach in the Ovsyannikov's spirit is no longer efficient when we attempt classifying PDEs with arbitrary functions of several variables.

Next, we compute the equivalence group $\mathcal{E}$ of equation (2.8). This group is generated by invertible transformations of the space V preserving the differential structure of equation (2.8) (see, e.g., [14]). Saying it another way, group transformation from $\mathcal{E}$

$$\bar{t} = \alpha(t, x, u), \qquad \bar{x} = \beta(t, x, u), \qquad v = U(t, x, u), \qquad \frac{D(\bar{t}, \bar{x}, v)}{D(t, x, u)} \neq 0,$$

should reduce (2.8) to equation of the same form

$$v_{\bar{t}\bar{t}} = v_{\bar{x}\bar{x}} + \tilde{g}(\bar{t},\bar{x},v)v_{\bar{x}} + \tilde{f}(\bar{t},\bar{x},v), \;\; \tilde{g}_v \neq 0$$

with possibly different $\tilde{f}, \tilde{g}$.

As proved by Ovsyannikov [14], it is possible to modify the Lie's infinitesimal approach to calculate equivalence group in essentially same way as invariance group. We omit the simple intermediate calculations and present the final result.

Assertion 2 *The maximal equivalence group $\mathcal{E}$ of equation (2.8) reads as*

$$\bar{t} = kt + k_1, \quad \bar{x} = \epsilon kx + k_2, \quad v = X(x)u + Y(t,x), \tag{3.1}$$

where $k \neq 0, \;\; X \neq 0, \;\; \epsilon = \pm 1, k, k_1, k_2 \in \mathbb{R}$, and X, Y are arbitrary smooth functions.

This completes the first step of the algorithm.

3.1 Preliminary group classification of equation (2.8).

First, select equations of the form (2.8) admitting one-parameter invariance groups.

Lemma 1 *There exist transformations (3.1) that reduce operator (2.4) to one of the six forms:*

$$\begin{aligned} &Q = m(t\partial_t + x\partial_x), \quad m \neq 0; \quad Q = \partial_t + \beta\partial_x, \quad \beta \geq 0;\\ &Q = \partial_t + \sigma(x)u\partial_u, \;\; \sigma \neq 0; \quad Q = \partial_x;\\ &Q = \sigma(x)u\partial_u, \quad \sigma \neq 0; \quad Q = \theta(t,x)\partial_u, \quad \theta \neq 0. \end{aligned} \tag{3.2}$$

Proof. Change of variables (3.1) reduces operator (2.4) to the form

$$\tilde{Q} = k(\lambda t + \lambda_1)\partial_{\bar{t}} + \epsilon k(\lambda x + \lambda_2)\partial_{\bar{x}} + [Y_t(\lambda t + \lambda_1) + (\lambda x + \lambda_2)(X'u + Y_x) + X(hu + r)]\partial_v. \tag{3.3}$$

If $\lambda \neq 0$ in (2.4), then putting $k_1 = \lambda^{-1}\lambda_1 k, \;\; k_2 = \epsilon\lambda^{-1}\lambda_2 k$, and taking as X, Y $(X \neq 0)$ integrals of system of PDEs

$$\begin{aligned} &X'(\lambda x + \lambda_2) + Xh = 0,\\ &Y_t(\lambda t + \lambda_1) + Y_x(\lambda x + \lambda_2) + Xr = 0 \end{aligned}$$

in (3.1), we reduce (3.3) to the form

$$\tilde{Q} = \lambda(\bar{t}\partial_{\bar{t}} + \bar{x}\partial_{\bar{x}}).$$

Provided $\lambda = 0$ and $\lambda_1 \neq 0$, we analogously obtain

$$\tilde{Q} = \partial_{\bar{t}} + \beta\partial_{\bar{x}}, \quad \beta \geq 0; \quad Q = \partial_{\bar{t}} + \sigma(\bar{x})v\partial_v, \quad \sigma \neq 0.$$

Next, if $\lambda = \lambda_1 = 0, \;\; \lambda_2 \neq 0$, in (2.4), then putting $k = \epsilon\lambda_2^{-1}$, and taking as X, Y $(X \neq 0)$ integrals of equations

$$\lambda_2 X' + hX = 0, \qquad Y_x + rX = 0,$$

we reduce operator (3.3) to become $\tilde{Q} = \partial_{\bar{x}}$.

Finally, the case $\lambda = \lambda_1 = \lambda_2 = 0$, gives rise to operators $\tilde{Q} = \sigma(\bar{x})v\partial_v, \qquad \tilde{Q} = \theta(\bar{t},\bar{x})\partial_v$.

Rewriting the above operators in the initial variables t, x completes the proof.

Theorem 2 *There are exactly five inequivalent equations of the form (2.8) that admit one-parameter transformation groups. They are listed below together with one-dimensional Lie algebras generating their invariance groups (note that we do not present the full form of invariant PDEs just the functions f and g)*

$$
\begin{aligned}
A_1^1 &= \langle t\partial_t + x\partial_x\rangle : g = x^{-1}\tilde{g}(\psi, u),\\
&\quad f = x^{-2}\tilde{f}(\psi, u), \psi = tx^{-1}, \tilde{g}_u \neq 0;\\
A_1^2 &= \langle \partial_t + \beta\partial_x\rangle : g = \tilde{g}(\eta, u),\ \ f = \tilde{f}(\eta, u),\\
&\quad \eta = x - \beta t, \quad \beta \geq 0,\ \ \tilde{g}_u \neq 0;\\
A_1^3 &= \langle \partial_t + \sigma(x)u\partial_u\rangle : g = -2\sigma'\sigma^{-1}\ln|u| + \tilde{g}(\rho, x),\\
&\quad f = (\sigma'\sigma^{-1})^2 u\ln^2|u| - \sigma'\sigma^{-1}\tilde{g}(\rho, x)u\ln|u| - \sigma^{-1}\sigma'' u\ln|u| + u\tilde{f}(\rho, x),\\
&\quad \rho = u\exp(-t\sigma), \quad \sigma \neq 0;\\
A_1^4 &= \langle \partial_x\rangle : g = \tilde{g}(t, u),\ \ f = \tilde{f}(t, u),\ \ \tilde{g}_u \neq 0;\\
A_1^5 &= \langle \sigma(x)u\partial_u\rangle : g = -2\sigma'\sigma^{-1}\ln|u| + \tilde{g}(t, x),\ \ f = (\sigma'\sigma^{-1})^2 u\ln^2|u|\\
&\quad -(\sigma^{-1}\sigma'' + \sigma^{-1}\sigma'\tilde{g}(t, x))u\ln|u| + u\tilde{f}(t, x), \quad \sigma' \neq 0.
\end{aligned}
$$

Proof. If equation (2.8) admits a one-parameter invariance group, then it is generated by operator of the form (2.4). According to Lemma 1, the latter is equivalent to one of the six operators (3.2). That is why, all we need to do is integrating six systems of determining equations corresponding to operators (2.6). For the first five operators solutions of determining equations are easily shown to have the form given in the statement of theorem.

We consider in more detail the operator $Q = \theta(t, u)\partial_u$. Determining equations (2.6) for this operator reduce to the form

$$\theta_{tt} - \theta_{xx} = \theta f_u + \theta_x g, \qquad \theta g_u = 0,$$

whence we get $g_u = 0$. Consequently, the system of determining equations is incompatible and the corresponding invariant equation fails to exist.

Non equivalence of the invariant equations follows from non equivalence of the corresponding symmetry operators.

The theorem is proved.

Lemma 2 *There are no realizations of semi-simple Lie algebras generated by operators of the form (2.4).*

Proof. To prove the lemma it suffices to check that there are no realizations of the lowest order simple Lie algebras by operators (2.4). The commutation relations defining those read as [37]:

$$
\begin{aligned}
&so(3) = \langle e_1, e_2, e_3\rangle :\ [e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1;\\
&sl(2, \mathbb{R}) = \langle e_1, e_2, e_3\rangle :\ [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1.
\end{aligned}
$$

We start by noting that one of the basis operators e_1, e_2, e_3 can be reduced to one of the five operators (3.2) (see, Lemma 1).

We consider in detail the case of operator

$$t\partial_t + x\partial_x \tag{3.4}$$

only, since the remaining operators are treated in a similar way.

Let the basis operator e_1 of the algebras $so(3)$ and $sl(2,\mathbb{R})$ be of the form (3.4). Computing the commutator of e_1 and Q of the form (2.4) yields the relation

$$[e_1, Q] = -\lambda_1\partial_t - \lambda_2\partial_x + [xh'u + xr_x + tr_t]\partial_u.$$

To satisfy the first two commutation relations for each of the algebras under study, the basis operators e_2, e_3 are to be of the form

$$\alpha_1\partial_t + \alpha_2\partial_x + (\gamma(x)u + \mu(t,x))\partial_u,$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$, γ and μ are smooth functions. It is straightforward to verify that these operators cannot satisfy the third commutation relation for either algebra $sl(2,\mathbb{R})$ and $so(3)$.

The lemma is proved.

Theorem 3 *There are no nonlinear equations (2.8) whose invariance algebras are isomorphic to semi-simple Lie algebras or contain those as sub-algebras.*

Proof. Suppose the inverse. Let (2.8) be an equation whose invariance algebra contain sub-algebra that is semi-simple Lie algebra L. Then by properties of semi-simple Lie algebras, there exist linear combinations of the basis elements of L forming the basis of either $so(3)$ or $sl(2,\mathbb{R})$. However, due to Lemma 2 there are no realizations of the algebras $so(3)$, $sl(2,\mathbb{R})$ by operators (2.4). We arrive at the contradiction which proves the theorem.

It follows from Theorem 3 and Levi-Maltsev theorem (see, e.g., [37, 38]) that nonlinear equations (2.8) can admit invariance algebras of the dimension higher than one, provided, (1) those are isomorphic to real solvable Lie algebras, or (2) their finite dimensional sub-algebras are real and solvable. Using this fact and also the concept of compositional row for solvable Lie algebras we may perform hierarchical classification of invariant equations starting from the lowest dimensional solvable Lie algebras and increasing dimension by one till we exhaust all possible invariant equations. We start by considering two-dimensional solvable Lie algebras.

There exist two inequivalent two-dimensional solvable Lie algebras [38, 39]

$$\begin{aligned} A_{2.1} &= \langle e_1, e_2\rangle \ : \ [e_1, e_2] = 0;\\ A_{2.2} &= \langle e_1, e_2\rangle \ : \ [e_1, e_2] = e_2. \end{aligned}$$

To construct all possible realizations of the above algebras we take as the first basis element one of the realizations of one-dimensional invariance algebras obtained above. The second operator is looked for in the form (2.4). In the case of commutative algebra $A_{2.1}$ there is no difference between operators e_1 and e_2, while for the algebra $A_{2.2}$ those two operators need separate analysis. We give full computation details for the case, when one of the basis elements is of the form A_1^1 given in Theorem 2.

Algebra $A_{2.1}$. Let operator e_1 be of the form (3.4) and operator e_2 read as (2.4). Then it follows from the relation $[e_1, e_2] = 0$ that $\lambda_1 = \lambda_2 = xh' = 0$, $tr_t + xr_x = 0$. Consequently, we may choose the basis elements of the algebra realization in the form $\langle t\partial_t + x\partial_x, (mu + r(\psi))\partial_u\rangle$, where $m \in \mathbb{R}$, $\psi = tx^{-1}$. Provided $m = 0$, the operator e_2 becomes $r(\psi)\partial_u$. As established earlier, this realization does not satisfy the determining equations. Hence, $m \neq 0$. Making the change of variables

$$\bar{t} = t, \quad \bar{x} = x, \quad v = u + m^{-1}r(\psi)$$

reduces the basis operators in question to the form $\bar{t}\partial_{\bar{t}} + \bar{x}\partial_{\bar{x}}$, $mv\partial_v$. That is why we can restrict our considerations to the realization $\langle t\partial_t + x\partial_x, u\partial_u\rangle$.

The second determining equation from (2.6) after being written for the operator $u\partial_u$ takes the form $ug_u = 0$, whence it follows that this realization does not satisfy the determining equations. So the realization A_1^1 cannot be extended to a realization of the two-dimensional algebra $A_{2.1}$.

Algebra $A_{2.2}$. If operator e_1 is of the form (3.4), then it follows from $[e_1, e_2] = e_2$ that $\lambda = \lambda_1 = \lambda_2 = 0$, $xh' = h$, $tr_t + xr_x = r$.

Next, if e_2 reads as (3.4), then we get from $[e_1, e_2] = e_2$ the erroneous equality $1 = 0$.

So the only possible case is when $e_2 = (mxu + xr(\psi))\partial_u$, $m \neq 0$, $\psi = tx^{-1}$, which gives rise to the following realization of the algebra $A_{2.2}$: $\langle t\partial_t + x\partial_x, xu\partial_u\rangle$. This algebra is indeed invariance algebra of an equation from the class (2.8) and the functions f and g read as: $g = -2x^{-1}\ln|u| + x^{-1}\tilde{g}(\psi)$, $f = x^{-2}u\ln^2|u| - x^{-2}\tilde{g}(\psi)u\ln|u| + x^{-2}u\tilde{f}(\psi)$, $\psi = tx^{-1}$.

Analysis of the remaining realizations of one-dimensional Lie algebras yields ten inequivalent $A_{2.1}$- and $A_{2.2}$-invariant equations (see the assertions below). What is more, the obtained (two-dimensional) algebras are maximal symmetry algebras of the corresponding equations.

Theorem 4 *There are, at most, four inequivalent $A_{2.1}$-invariant nonlinear equations (2.8). Below we list the realizations of $A_{2.1}$ and the corresponding expressions for f and g.*

1) $\langle \partial_t, \sigma(x)u\partial_u, \rangle : g = -2\sigma'\sigma^{-1}\ln|u|,$

$$f = (\sigma'\sigma^{-1})^2 u\ln^2|u| - \sigma^{-1}\sigma' u\ln|u| + u\tilde{f}(x), \quad \sigma' \neq 0;$$

2) $\langle \partial_t, \partial_x\rangle : g = \tilde{g}(u),\ f = \tilde{f}(u),\ \tilde{g}_u \neq 0;$

3) $\langle \partial_x, \partial_t + u\partial_u\rangle : g = \tilde{g}(\omega),\ f = \exp(t)\tilde{f}(\omega),\ \omega = \exp(-t),\ \tilde{g}_\omega \neq 0;$

4) $\langle \sigma(x)u\partial_u, \partial_t - \frac{1}{2}k\sigma(x)\psi(x)u\partial_u\rangle : g = -2\sigma'\sigma^{-1}\ln|u| + kt + \tilde{g}(x),$

$$f = (\sigma'\sigma^{-1})^2 u\ln^2|u| - \sigma^{-1}\sigma'' u\ln|u| - \sigma^{-1}\sigma'(kt + \tilde{g}(x))u\ln|u|$$

$$+u\left[\frac{1}{2}k\sigma'\sigma^{-1}t + \frac{1}{4}k^2t^2 + \frac{1}{2}k\tilde{g}(x) + \tilde{f}(x)\right],$$

$$k \neq 0,\ \sigma' \neq 0,\ \psi = \int \sigma^{-1}dx.$$

Theorem 5 *There exist, at most, six inequivalent $A_{2.2}$-invariant nonlinear equations (2.8). Below we list the realizations of $A_{2.1}$ and the corresponding expressions for f and g.*

1) $\langle t\partial_t + x\partial_x, k^{-1}|x|^k u\partial_u\rangle : g = x^{-1}(-2k\ln|u| + \tilde{g}(\psi)),$
$f = x^{-2}u(-k^2\ln^2|u| + k\tilde{g}(\psi)\ln|u| + k(k-1)\ln|u| + \tilde{f}(\psi)),$
$k \neq 0,\ \psi = tx^{-1};$

2) $\langle \partial_t + \beta\partial_x,\ \exp(\beta^{-1}x)u\partial_u\rangle : g = -2\beta^{-1}\ln|u| + \tilde{g}(\eta),$
$f = \beta^{-2}u\ln^2|u| - (\beta^{-2} + \beta^{-1}\tilde{g}(\eta))u\ln|u| + u\tilde{f}(\eta),$
$\beta > 0,\ \eta = x - \beta t;$

3) $\langle -t\partial_t - x\partial_x, \partial_t + \beta\partial_x\rangle : g = \eta^{-1}\tilde{g}(u),\quad f = \eta^{-2}\tilde{f}(u),\ \beta \geq 0,$
$\eta = x - \beta t,\ \tilde{g}_u \neq 0;$

4) $\langle -t\partial_t - x\partial_x, \partial_t + mx^{-1}u\partial_u\rangle : g = x^{-1}(2m\psi + \tilde{g}(\omega)),$
$f = x^{-1}[-2m\psi u - 2m\psi - 2 - \tilde{g}(\omega) + \exp(m\psi)\tilde{g}(\omega)],$
$m > 0,\ \omega = u\exp(-m\psi),\ \psi = tx^{-1},\ \tilde{g}_\omega \neq 0;$

5) $\langle \partial_x, e^x u\partial_u\rangle : g = -2\ln|u| + \tilde{g}(t),\ f = u\ln^2|u| -$
$-u\ln|u|(1 + \tilde{g}(t)) + u\tilde{f}(t);$

6) $\langle -t\partial_t - x\partial_x, \partial_x\rangle : g = t^{-1}\tilde{g}(u),\ f = t^{-2}\tilde{f}(u),\ \tilde{g}_u \neq 0.$

3.2 Completing group classification of (2.8).

As the invariant equations obtained in the previous subsection contain arbitrary functions of, at most, one variable, we can use the standard Lie-Ovsyannikov approach to complete group classification of (2.8). We give the computation details for the case of the first $A_{2.1}$-invariant equation, the remaining cases are treated in a similar way.

Putting $g = -2\sigma'\sigma^{-1}\ln|u|, f = (\sigma'\sigma^{-1})u\ln^2|u| - \sigma^{-1}\sigma''u\ln|u| + u\tilde{f}(x),\ \sigma = \sigma(x),\ \sigma' \neq 0$ we rewrite the first determining equation to become:

$$-2h' + 2\lambda\sigma'\sigma^{-1}\ln|u| = -2(\lambda x + \lambda_2)(\sigma'\sigma^{-1})'_x\ln|u| - 2h\sigma'\sigma^{-1} - 2r\sigma'\sigma^{-1}u^{-1}.$$

As $h = f(x), \sigma = \sigma(x),\ r = r(t,x), \lambda, \lambda_2 \in \mathbb{R}$, the above relation is equivalent to the following ones:

$$h' = \sigma'\sigma^{-1}h, \quad r = 0, \quad \lambda\sigma'\sigma^{-1} = -(\lambda x + \lambda_2)(\sigma'\sigma^{-1})'.$$

If σ is an arbitrary function, then $\lambda = \lambda_2 = r = 0,\ h = C\sigma,\ C \in \mathbb{R}$ and we get $\langle\partial_t, \sigma(x)u\partial_u\rangle$ as the maximal symmetry algebra. Hence, the extension of symmetry algebra is only possible when the function $\psi = \sigma'\sigma^{-1}$ is a (non-vanishing identically) solution of equation

$$(\alpha x + \beta)\psi' + \alpha\psi = 0, \quad \alpha, \beta \in R,\ |\alpha| + |\beta| \neq 0.$$

If $\alpha \neq 0$, then at the expense of displacements by x we can get $\beta = 0$, so that $\psi = mx^{-1},\ m \neq 0$. Integrating the remaining determining equations we get

$$g = -2mx^{-1}\ln|u|, \quad f = mx^{-2}[mu\ln^2|u| - (m-1)u\ln|u| + nu], \quad m \neq 0,\ m, n \in \mathbb{R}.$$

The maximal invariance algebra of the obtained equation is the three-dimensional Lie algebra $\langle\partial_t, |x|^m u\partial_u, t\partial_t + x\partial_x\rangle$ isomorphic to $A_{3.7}$.

Next, if $\alpha = 0$, then $\beta \neq 0$ and $\psi = m,\ m \neq 0$. If this is the case, we have

$$g = \ln|u|,\ f = \frac{1}{4}u\ln^2|u| - \frac{1}{4}u\ln|u| + nu,\ n \in R.$$

The maximal invariance algebra of the above equation reads as

$$\langle\partial_t, \partial_x, \exp\left(-\frac{1}{2}x\right)u\partial_u\rangle$$

and is isomorphic to $A_{3.2}$.

Similarly we prove that the list of inequivalent equations of the form (2.8) admitting three-dimensional symmetry algebras is exhausted by the equations given below. Note that the presented algebras are maximal. This means, in particular, that maximal symmetry algebra of equation (2.8) is, at most, three-dimensional.

$A_{3.2}$-invariant equations

1) $u_{tt} = u_{xx} + u_x\ln|u| + \frac{1}{4}u\ln^2|u| - \frac{1}{4}u\ln|u| + nu\ (n \in \mathbb{R}):\ \langle\partial_t, \partial_x, \exp\left(-\frac{1}{2}x\right)u\partial_u\rangle;$

2) $u_{tt} = u_{xx} + m[\ln|u| - t]u_x + \frac{m^2}{4}u[(\ln|u| - t)(\ln|u| - t - 1)] + nu(m > 0,$
$n \in R) : \langle\partial_x, \partial_t + u\partial_u, \exp\left(-\frac{1}{2}mx\right)u\partial_u\rangle.$

$A_{3.4}$-invariant equations

1) $u_{tt} = u_{xx} + x^{-1}[2\ln|u| + mx^{-1}t + n]u_x + x^{-2}u\ln|u|$
$+(mx^{-1}t + n - 2)x^{-2}u\ln|u| + \frac{1}{4}m^2x^{-4}t^2u + \frac{1}{2}m(n-3)x^{-3}tu + px^{-2}u$
$(\ m \neq 0,\ n, p \in \mathbb{R}) : \langle t\partial_t + x\partial_x, x^{-1}u\partial_u, \partial_t - \frac{m}{2}x^{-1}\ln|x|u\partial_u\rangle.$

$A_{3.5}$-invariant equations

$$1)\ u_{tt} = u_{xx} + |u|^m u_x + n|u|^{1+2m}\ (m \neq 0,\ n \in \mathbb{R}) : \langle \partial_t, \partial_x, t\partial_t + x\partial_x - m^{-1}u\partial_u \rangle;$$

$$2)\ u_{tt} = u_{xx} + e^u u_x + ne^{2u}\ (n \in \mathbb{R}) : \langle \partial_t, \partial_x, t\partial_t + x\partial_x - \partial_u \rangle;$$

$$3)\ u_{tt} = u_{xx} - x^{-1}[2\ln|u| - mx^{-1}t - n]u_x + x^{-2}u\ln^2|u|$$
$$-x^{-2}(mx^{-1}t + n)u\ln|u| + ux^{-2}\left[\tfrac{m}{4}x^{-2}t^2 + \tfrac{m}{2}(n-1)x^{-1}t + p\right]$$
$$(m, n, p \in \mathbb{R}) : \langle t\partial_t + x\partial_x, xu\partial_u, \partial_t + \tfrac{m}{4}x^{-1}u\partial_u \rangle.$$

$A_{3.7}$-invariant equations

$$1)\ u_{tt} = u_{xx} - 2mx^{-1}u_x\ln|u| + mx^{-2}[mu\ln^2|u| - (m-1)u\ln|u| + nu]$$
$$(m \neq 0, 1; n \in \mathbb{R}) : \langle \partial_t, |x|^m u\partial_u, t\partial_t + x\partial_x \rangle;$$

$$2)\ u_{tt} = u_{xx} - x^{-1}[2k + \ln|u| - mx^{-1}t - n]u_x + k^2x^{-2}u\ln^2|u|$$
$$-kx^{-2}[mtx^{-1} + k + n - 1]u\ln|u| + \tfrac{1}{2}m(k - 2 + n)tx^{-3}u$$
$$+\tfrac{1}{4}m^2t^2x^{-4}u + px^{-2}u\quad (|k| \neq 0, 1; m \neq 0, n, p \in \mathbb{R}) :$$
$$\langle t\partial_t + x\partial_x, |x|^k u\partial_u, \partial_t + \tfrac{m}{2(1+k)}x^{-1}u\partial_u \rangle.$$

This completes group classification of nonlinear equations (2.8).

4 Group classification of equation (2.12)

Omitting calculation details we present below the determining equations for symmetry operators admitted by equation (2.12).

Assertion 3 *The maximal invariance group of PDE (2.12) is generated by the infinitesimal operator*

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + [h(t)u + r(t,x)]\partial_u, \tag{4.1}$$

where τ, ξ, h, r, f, g *are smooth functions satisfying the conditions*

$$\begin{aligned} &r_{tx} + f[h - \tau_t - \xi_x] = gr_x + \tau f_t + \xi f_x + [hu + r]f_u, \\ &h_t = \tau_t g + \tau g_t + \xi g_x. \end{aligned} \tag{4.2}$$

Assertion 4 *The equivalence group* $\mathcal{E}$ *of (2.12) is formed by the following transformations of the space* V*:*

$$\begin{aligned} &(1) \quad \bar{t} = T(t),\ \bar{x} = X(x),\ v = U(t)u + Y(t,x),\quad t'X'U \neq 0; \\ &(2) \quad \bar{t} = T(x),\ \bar{x} = X(t),\ v = \Psi(x)\Phi(t,x)u + Y(t,x),\quad t'X'\Psi \neq 0, \\ &\qquad \Phi(t,x) = \exp[-\int g(t,x)dt],\ g_x \neq 0. \end{aligned} \tag{4.3}$$

As the direct verification shows, given arbitrary functions g and f, it follows from (4.2) that $\tau = h = \xi = r = 0$. So that in the generic case the maximal invariance group of (2.12) is the trivial group of identical transformations.

We begin classification of (2.12) by constructing equations that admit one-dimensional symmetry algebras.

Lemma 3 *There exist transformations (4.3) reducing operator (4.1) to one of the seven canonical forms given below*

$$\begin{aligned} Q &= t\partial_t + x\partial_x; \quad Q = \partial_t; \quad Q = \partial_x + tu\partial_u; \\ Q &= \partial_x + \epsilon u\partial_u, \ \epsilon = 0, 1; \ Q = tu\partial_u, \\ Q &= u\partial_u, \quad Q = r(t,x)\partial_u, \ r \neq 0. \end{aligned} \tag{4.4}$$

Proof. Transformations (4.3) reduce operator Q (4.1) to become

$$\tilde{Q} = \tau T'\partial_{\bar{t}} + \xi X'\partial_{\bar{x}} + [(\tau U' + Uh)u + \tau Y_t + \xi Y_x + Ur]\partial_v. \tag{4.5}$$

Provided $\sigma \cdot \xi \neq 0$, we can choose as T, X, U, Y non vanishing identically solutions of equations

$$\tau T' = T, \quad \xi X' = X, \quad \tau U' + hU = 0, \quad \tau Y_t + \xi Y_x + Ur = 0$$

thus getting operator $\tilde{Q}$ (4.5) in the form $\tilde{Q} = \bar{t}\partial_{\bar{t}} + \bar{x}\partial_{\bar{x}}$. If $\tau \neq 0$, a $\xi = 0$, then taking as T, U, Y solutions of equations

$$\tau T' = 1, \ \tau U' + hU = 0 \ (U \neq 0), \ \tau Y_t + Ur = 0$$

reduces operator (4.1) to become $\tilde{Q} = \partial_{\bar{t}}$. If $\tau = 0$, $\xi \neq 0$, then under $h' \neq 0$ we get operator $\tilde{Q} = \partial_{\bar{x}} + \bar{t}v\partial_v$. Next, if $h' = 0$, we arrive at the operator $\tilde{Q} = \partial_{\bar{x}} + \epsilon v\partial_v$, where either $\epsilon = 0$ or $\epsilon = 1$.

Finally, the case $\tau = \xi = 0$, gives rise to the following operators $\tilde{Q} = \bar{t}v\partial_v$, $\tilde{Q} = v\partial_v$, $\tilde{Q} = r(\bar{t}, \bar{x})\partial_v$. After rewriting these in the initial variables we get the operators listed in the statement of lemma. The lemma is proved.

Theorem 6 *There exist, at most, three inequivalent nonlinear equations (2.12) that admit one-dimensional invariance algebras. The form of functions f, g and the corresponding symmetry algebras are given below.*

$$\begin{aligned} A_1^1 &= \langle t\partial_t + x\partial_x \rangle : g = t^{-1}\tilde{g}(\omega), \ f = t^{-2}f(u,\omega), \quad \omega = tx^{-1}, \quad \tilde{g}_\omega \neq 0, \quad f_{uu} \neq 0; \\ A_1^2 &= \langle \partial_t \rangle : g = \tilde{g}(x), \ f = \tilde{f}(x,u), \ \tilde{g}' \neq 0, \ \tilde{f}_{uu} \neq 0; \\ A_1^3 &= \langle \partial_x + tu\partial_u \rangle : g = x + \tilde{g}(t), f = e^{tx}\tilde{f}(t,\omega), \ \omega = e^{-tx}u, \ \tilde{f}_{\omega\omega} \neq 0. \end{aligned}$$

Proof. If equation (2.12) admits one-parameter transformation group, then the latter is generated by infinitesimal operator (4.1). According to Lemma 3 there exist equivalence transformations (4.3) reducing this operator to one of the seven canonical operators (4.4). With this fact in hand, we turn to solving the determining equations (4.2) for each of those operators. The first three operators yield invariant equations and corresponding symmetry algebras given in the statement of theorem. The next two operators give rise to inconsistent equations.

Finally, the remaining operators yield that the functions f and g are linear in u, which means that the corresponding invariant equations are linear.

It is straightforward to verify that for the case of arbitrary functions $\tilde{f}$, $\tilde{g}$, the corresponding one-dimensional algebras are maximal invariance algebras.

The theorem is proved.

We proceed now to analyzing equations (2.12) admitting two-dimensional symmetry algebras.

Theorem 7 *There exist, at most, three inequivalent nonlinear equations (2.12) that admit two-dimensional symmetry algebras, all of them being $A_{2.2}$-invariant equations. The forms of functions f and g*

and the corresponding realizations of the Lie algebra $A_{2.2}$ are given below

$$
\begin{array}{rcl}
A^1_{2.2} & = & \langle t\partial_t + x\partial_x, t^2\partial_t + x^2\partial_x + mut\partial_u\rangle \ (m \in \mathbb{R}) : \\
 & & g = [mt + (k-m)x]t^{-1}(t-x)^{-1}, \ k \neq 0, \\
 & & f = |t-x|^{m-2}|x|^{-m}\tilde{f}(\omega), \\
 & & \omega = u|t-x|^{-m}|x|^m, \tilde{f}_{\omega\omega} \neq 0; \\
A^2_{2.2} & = & \langle t\partial_t + x\partial_x, t^2\partial_t + mtu\partial_u\rangle \ (m \in \mathbb{R}) : \\
 & & g = t^{-2}[kx + mt], \ k \neq 0, \ f = |t|^{m-2}|x|^{-m}\tilde{f}(\omega), \\
 & & \omega = |t|^{-m}|x|^m u, \ \tilde{f}_{\omega\omega} \neq 0; \\
A^3_{2.2} & = & \langle t\partial_t + x\partial_x, x^2\partial_x + tu\partial_u\rangle : \\
 & & g = (tx)^{-1}(mx - t) \ (m \in \mathbb{R}), \ f = x^{-2}\exp(-tx^{-1})\tilde{f}(\omega), \\
 & & \omega = u\exp(tx^{-1}), \ \tilde{f}_{\omega\omega} \neq 0.
\end{array}
$$

To prove the theorem we need to extend realizations A^i_1 $(i = 1, 2, 3)$ to realizations of the algebras $A_{2.1}, A_{2.2}$ by operators (4.1). We skip calculation details.

Note that if the functions $\tilde{f}$ are arbitrary, then the invariance algebras given in the statement of Theorem 7 are maximal.

Now we can complete the group classification presented in Theorem 7 with the use of Lie-Ovsyannikov classification routine.

We consider in some detail the case of $A^1_{2.2}$-invariant equations (the remaining cases are treated in a similar way). The second determining equation from (4.2) reads now as

$$(t-x)^2h_t = t^{-1}\tau_t[m(t-x)^2 + kx(t-x)] \quad + \tau[-t^{-2}m(t-x)^2 - 2kt^{-1}x + kt^{-2}x^2] + k\xi. \tag{4.6}$$

Differentiating right- and left-hand sides of (4.6) twice by x yields

$$h_t = (m-k)(t^{-1}\tau_t - t^{-2}\tau) + k\xi''.$$

Hence we get $\xi''' = 0$ and

$$
\begin{array}{rcl}
\xi & = & \lambda_1x^2 + \lambda_2x + \lambda_3, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, \\
h & = & (m-k)t^{-1}\tau + \lambda_1kt + \lambda_4, \quad \lambda_4 \in \mathbb{R}.
\end{array}
$$

With account of the above results we obtain from (4.6) that $\tau = \lambda_1t^2+\lambda_2t+\lambda_3$. So it follows from (4.6) that the coefficients of infinitesimal operator (4.1), which generates symmetry group of $A^1_{2.2}$-invariant equation, read necessarily as

$$
\begin{array}{rcl}
\tau & = & \lambda_1t^2 + \lambda_2t + \lambda_3, \\
\xi & = & \lambda_1x^2 + \lambda_2x + \lambda_3, \\
h & = & m\lambda_1t + (m-k)\lambda_3t^{-1} + (m-k)\lambda_2 + \lambda_4, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}.
\end{array}
$$

Consequently, the first determining equation from (4.2) takes the form

$$
\begin{array}{l}
\{x^{-1}(t-x)^{-1}[((m-k)\lambda_2 + \lambda_4)(tx - x^2) - (m-k)\lambda_3t^{-1}x^2 \\
-k\lambda_3x + m\lambda_3t]\omega + r|t-x|^{-m}|x|^m\}\tilde{f}_\omega \\
-x^{-1}(t-x)^{-1}[((m-k)\lambda_2 + \lambda_4)(tx - x^2) \\
-(m-k)\lambda_3t^{-1}x^2 - k\lambda_3x + m\lambda_3t]\tilde{f} \\
= |t-x|^{-m+2}|x|^m[r_{tx} - t^{-1}(m + kx(t-x)^{-1}r_x].
\end{array} \tag{4.7}
$$

It follows from (4.7) that if the functions$\tilde{f}$ are arbitrary, then the maximal invariance algebra of the equation under study coincide with the realization $A^1_{2.2}$. What is more, an extension of the invariance algebra is only possible when the function $\tilde{f}$ obey the following equation:

$$(a\omega + b)\tilde{f}_\omega - a\tilde{f} = c, \tag{4.8}$$

where $a, b, c \in R$, $|a| + |b| \neq 0$. On the other hand, it follows from (4.8)

$$(a\omega + b)\tilde{f}_{\omega\omega} = 0,$$

whence $f_{\omega\omega} = 0$. We arrive at the contradiction, which proves that there are no extension of the realization $A^1_{2.2}$ in question to the higher dimensional invariance algebra of the equation (2.12). Analyzing $A^2_{2.2}$- and $A^2_{2.3}$-invariant equations we arrive at the same conclusion.

Consequently, there are no nonlinear equations of the form (2.12) whose maximal invariance algebras are solvable Lie algebras of the dimension higher than two. Next, as the algebra $sl(2,\mathbb{R})$ contains two-dimensional subalgebra isomorphic to $A_{2.2}$, there are no nonlinear equations (2.12), whose invariance algebras are either isomorphic to $sl(2,\mathbb{R})$ or contain it as a subalgebra. Finally, we verified that there are no realizations of the algebra $so(3)$ by operators (4.1).

Summing up the above reasonings we formulate the following assertion.

Theorem 8 *A nonlinear equation (2.12) having non-trivial symmetry properties is equivalent to one of the equations listed in Theorems 6 and 7.*

This completes group classification of the class of nonlinear PDEs (2.12).

5 Group classification of equation (2.13)

As earlier, we present the results of the first step of our group classification algorithm skipping derivation details.

Assertion 5 *Invariance group of equation (2.13) is generated by infinitesimal operator*

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + (ku + r(t,x))\partial_u, \tag{5.1}$$

where k is a constant τ, ξ, r, f are functions satisfying the relation

$$r_{tx} + [k - \tau' - \xi']f = \tau f_t + \xi f_x + [ku + r]f_u. \tag{5.2}$$

Assertion 6 *Equivalence group $\mathcal{E}$ of the class of equations (2.13) is formed by the following transformations:*

$$\begin{array}{ll} (1) & \bar{t} = T(t), \quad \bar{x} = X(x), \quad v = mu + Y(t,x), \\ (2) & \bar{t} = T(x), \quad \bar{x} = X(t), \quad v = mu + Y(t,x), \quad T'X'm \neq 0. \end{array} \tag{5.3}$$

Note that given arbitrary f, it follows from (5.2) that $\tau = \xi = k = r = 0$, i.e., the group admitted is trivial. To obtain equations with nontrivial symmetry we need to specify properly the function f. To this end we perform classification of equations under study admitting one-dimensional invariance algebras.

Lemma 4 *There exist transformations from the group $\mathcal{E}$ (5.3) that reduce (5.1) to one of the four canonical forms:*

$$\begin{aligned} Q &= \partial_t + \partial_x + \epsilon u\partial_u \quad (\epsilon = 0,1): \\ Q &= \partial_t + \epsilon u\partial_u \quad (\epsilon = 0,1); \\ Q &= u\partial_u, \ Q = g(t,x)\partial_u \ (g \neq 0). \end{aligned}$$

Proof. Utilizing transformations (1) from (5.3) we reduce the operator Q to one of the following forms:

$$\begin{aligned}
Q &= \partial_{\bar{t}} + \partial_{\bar{x}} + \epsilon v\partial_v \quad (\epsilon = 0, 1) : \\
Q &= \partial_{\bar{t}} + \epsilon v\partial_v \quad (\epsilon = 0, 1); \\
Q &= \partial_{\bar{x}} + \epsilon v\partial_v \quad (\epsilon = 0, 1); \\
Q &= v\partial_v, \ Q = g(\bar{t}, \bar{x})\partial_v \ (g \neq 0).
\end{aligned}$$

Next, we note that the change of variables $\tilde{t} = \bar{x}, \quad \tilde{x} = \bar{t}, \quad \tilde{v} = v$, which is of the form (2) from (5.3), transforms the second operator into the third one.

Finally, rewriting the obtained operators in the initial variables completes the proof.

Theorem 9 *There exist exactly two nonlinear equations of the form (2.13) admitting one-dimensional invariance algebras. The corresponding expressions for function f and invariance algebras are given below.*

$$\begin{aligned}
A_1^1 &= \langle \partial_t + \partial_x + \epsilon u\partial_u \rangle \ (\epsilon = 0, 1) : \ f = e^{\epsilon t}\tilde{f}(\theta, \omega), \quad \theta = t - x, \ \omega = e^{-\epsilon t}u; \, \tilde{f}_{\omega\omega} \neq 0; \\
A_1^2 &= \langle \partial_t + \epsilon u\partial_u \rangle \ (\epsilon = 0, 1) : f = e^{\epsilon t}\tilde{f}(x, \omega), \quad \omega = e^{-\epsilon t}u, \ \tilde{f}_{\omega\omega} \neq 0.
\end{aligned}$$

To prove the theorem, it suffices to select those operators from the list of Lemma 4 that can be invariance algebra of nonlinear equation of the form (2.13). To this end we need to solve equation (5.2) for each of the operators in question.

The first two operators yield A_1^1- and A_1^2-invariant equations. The last two operators gives rise to linear invariant equations (2.13), which are not taken into account.

What is more, if the function $\tilde{f}$ is arbitrary, then the algebras A_1^1 and A_1^2 are maximal invariance algebras of the corresponding equations.

Next, we classify nonlinear equations admitting symmetry algebras of the dimension higher than one. We begin by considering equations whose invariance algebras contain semi-simple subalgebras. It turns out, that the class of operators (5.1) contain no realizations of the algebra $so(3)$. Furthermore, it contains four inequivalent realizations of the algebra $sl(2, \mathbb{R})$ given below.

$$\begin{aligned}
&(1) \qquad \langle \partial_t, \frac{1}{2}e^{2t}\partial_t, -\frac{1}{2}e^{-2t}\partial_t \rangle; \\
&(2) \qquad \langle \partial_t, \frac{1}{2}e^{2t}(\partial_t + \partial_u), -\frac{1}{2}e^{-2t}(\partial_t - \partial_u) \rangle; \\
&(3) \qquad \langle \partial_t, \frac{1}{2}e^{2t}(\partial_t + x\partial_u), -\frac{1}{2}e^{-2t}(\partial_t - x\partial_u) \rangle; \\
&(4) \qquad \langle \partial_t + \partial_x, \frac{1}{2}e^{2t}\partial_t + \frac{1}{2}e^{2x}\partial_x, -\frac{1}{2}e^{-2t}\partial_t - \frac{1}{2}e^{-2x}\partial_x + \epsilon[e^{-2x} - e^{-2t}]\partial_u \rangle, \quad \epsilon = 0, 1.
\end{aligned}$$

Before analyzing $sl(2, \mathbb{R})$-invariant equations let us briefly review the group properties of the Liouville equation

$$u_{tx} = \lambda e^u, \quad \lambda \neq 0. \tag{5.4}$$

It is a common knowledge that the maximal invariance group of this equation is the infinite-parameter group generated by the following infinitesimal operator [40]:

$$Q = h(t)\partial_t + g(x)\partial_x - (h' + g')\partial_u,$$

where h and g are arbitrary smooth functions. Note that due to this fact the Liouville equation can be linearized by a (non-local) change of variables (see, e.g., [11, 41, 42]).

After a simple algebra we obtain that realizations (1), (3), (4) with $\epsilon = 1$ cannot be invariance algebras of nonlinear equation of the form (2.13). Realization (2) is the invariance algebra of equation

$$u_{tx} = \tilde{f}(x)e^{-2u}, \quad \tilde{f} \neq 0$$

which reduces to equation (5.4) via the change of variables

$$t = t, \quad x = x, \quad u = -\frac{1}{2}(v - \ln|\tilde{f}|), \quad v = v(t, x).$$

Finally making use of the change of variables

$$\bar{t} = e^{-2t}, \quad \bar{x} = e^{-2x}, \quad v = u$$

we rewrite (4) under $\epsilon = 0$ to become

$$\langle \partial_t + \partial_x, t\partial_t + x\partial_x, t^2\partial_t + x^2\partial_x \rangle.$$

The corresponding invariant equation reads as

$$u_{tx} = (t - x)^{-2}\tilde{f}(u), \quad \tilde{f}_{uu} \neq 0. \tag{5.5}$$

If the function $\tilde{f}$ is arbitrary, then the above presented realization is the maximal invariance algebra of the equation under study. Using Lie-Ovsyannikov algorithm we establish that extension of symmetry is only possible when $\tilde{f} = \lambda e^u + 2$. However, the corresponding equation is reduced to the Liouville equation by the change of variables

$$t = t, \ x = x, \quad u = v(t, x) + 2\ln|t - x|.$$

Thus the only inequivalent nonlinear equations (2.13) whose invariance algebras contain semi-simple subalgebras are given in (5.4) and (5.5), where $\tilde{f}$ is an arbitrary smooth function of u.

To complete group classification of equation (2.13) we need to describe equations whose invariance algebras are solvable Lie algebras of the dimension higher than one. We begin with those realizations of two-dimensional Lie algebras $A_{2.1}$, $A_{2.2}$, which can be admitted by nonlinear equations (2.13).

It turns out that the class of operators (5.1) contains within the equivalence relation only one realization of the algebra $A_{2.1}$ which meets the invariance requirements, namely,

$$\langle \partial_t + \epsilon_1 u\partial_u, \partial_x + \epsilon_2 u\partial_u \rangle \ (\epsilon_1 = 0, 1; \quad \epsilon_2 = 0, 1).$$

The corresponding invariant equation reads as

$$u_{tx} = \exp(\epsilon_1 t + \epsilon_2 x)\tilde{f}(\omega), \quad \omega = u\exp(-\epsilon_1 t - \epsilon_2 x). \tag{5.6}$$

Analysis of equation (5.6) with arbitrary $f(\omega)$ shows that under $\epsilon_1 + \epsilon_2 \neq 0$ the above realization is its maximal invariance algebra. Provided $\epsilon_1 = \epsilon_2 = 0$, the equation takes the form

$$u_{tx} = f(u) \tag{5.7}$$

and its maximal invariance algebra is the three-dimensional Lie algebra of the operators

$$\langle \partial_t, \partial_x, t\partial_t - x\partial_x \rangle,$$

which is isomorphic to $A_{3.6}$.

It is a common knowledge (see, e.g., [17, 18, 19]) that (5.7) admits higher symmetry if it is equivalent either to the Liouville equation (5.4), or to the nonlinear d'Alembert equation

$$u_{tx} = \lambda|u|^{n+1}, \quad \lambda \neq 0, \quad n \neq 0, -1. \tag{5.8}$$

The maximal invariance algebra of (5.8) is the four-dimensional Lie algebra of the operators

$$\langle t\partial_t - \frac{1}{n}u\partial_u, x\partial_x - \frac{1}{n}u\partial_u, \partial_t, \partial_x\rangle.$$

It is isomorphic to the Lie algebra $A_{2.2} \oplus A_{2.2}$.

Extension of symmetry algebra of equation (5.6) with $\epsilon_1 = 1, \epsilon_2 = 0$, is only possible when:

$$u_{tx} = \lambda e^{-mt}|u|^{m+1}, \ \lambda \neq 0, \ m \neq 0, -1; \tag{5.9}$$
$$u_{tx} = \lambda e^{t}\exp(ue^{-t}), \ \lambda \neq 0. \tag{5.10}$$

The maximal invariance algebra of (5.9) is the four-dimensional Lie algebra of operators

$$\langle \partial_t + u\partial_u, e^{mt}\partial_t, \partial_x, x\partial_x - \frac{1}{m}u\partial_u\rangle,$$

which is isomorphic to $A_{2.2} \oplus A_{2.2}$. Note that the change of variables

$$\bar{t} = e^{-mt}, \ \bar{x} = x, \ u = v(\bar{t}, \bar{x})$$

reduces the above equation to the form (5.8).

The maximal invariance algebra of (5.10) is spanned by the operators

$$\langle \partial_t + u\partial_u, \partial_x, x\partial_x - e^{t}\partial_u\rangle,$$

and is isomorphic to $A_1 \oplus A_{2.2}$.

Analysis of $A_{2.2}$-invariant equations yields the following results. The class of operators (5.1) contains six inequivalent realizations of the algebra $A_{2.2}$ which meet the invariance requirements

$$\begin{array}{ll} (1) & \langle -t\partial_t + x\partial_u, \partial_t\rangle; \\ (2) & \langle -t\partial_t - x\partial_x, \partial_t + \partial_x\rangle; \\ (3) & \langle -t\partial_t - x\partial_x + u\partial_u, \partial_t + \partial_x\rangle; \\ (4) & \langle -t\partial_t + \partial_u, \partial_t\rangle; \\ (5) & \langle -t\partial_t - x\partial_x - u\partial_u, \partial_t\rangle; \\ (6) & \langle -t\partial_t - x\partial_x, \partial_t\rangle. \end{array} \tag{5.11}$$

Equation invariant under realization (1) reads as

$$u_{tx} = \exp(x^{-1}u). \tag{5.12}$$

Its maximal symmetry algebra is the three-dimensional Lie algebra of operators

$$\langle -t\partial_t + x\partial_u, \partial_t, x\partial_x + u\partial_u\rangle$$

isomorphic to $A_{2.2} \oplus A_1$. Note that the change of variables

$$\bar{t} = x, \quad \bar{x} = e^{t}, \quad u = v(\bar{t}, \bar{x})$$

reduces (5.12) to the form (5.10).

Equation invariant under the second realization of $A_{2.2}$ is of the form (5.5). It has already been studied while describing $sl(2,\mathbb{R})$-invariant equations.

Realizations (3) and (4) give no new invariant equations as well.

New invariant equation are obtained with the use of the fifth realization from (5.10). It has the form

$$u_{tx} = x^{-1}\tilde{f}(\omega), \quad \omega = x^{-1}u.$$

If the function $\tilde{f}$ is arbitrary, then the realization in question is maximal invariance algebra of the above equation. Further extension of symmetry properties is only possible if $\tilde{f}(\omega) = \lambda|\omega|^{m+1}$, which gives the following invariant equation:

$$u_{tx} = \lambda|x|^{-m-2}|u|^{m+1}, \quad \lambda \neq 0, \quad m \neq 0, -1, -2.$$

Its maximal symmetry algebra is the three-dimensional Lie algebra having the basis

$$\langle \partial_t, t\partial_t + x\partial_x + u\partial_u, x\partial_x + \frac{m+1}{m}u\partial_u \rangle.$$

This algebra is isomorphic to $A_{2.2} \oplus A_1$.

We sum up the above results in the following assertion.

Theorem 10 *The Liouville equation $u_{tx} = \lambda e^u, \quad \lambda \neq 0$, has the highest symmetry among equations (2.13). Its maximal invariance algebra is infinite-dimensional and is spanned by the following infinite set of basis operators:*

$$Q = h(t)\partial_t + g(x)\partial_x - (h'(t) + g'(x))\partial_u,$$

where h and g are arbitrary smooth functions. Next, there exist exactly nine inequivalent equations of the form (2.13), whose maximal invariance algebras have dimension higher that one. Those equations and their invariance algebras are given in Table 1.

Table I. *Invariant equations (6.9)*

Number	Function f	Symmetry operators	Invariance algebra type
1	$e^t\tilde{f}(\omega)$, $\omega = ue^{-t}, \tilde{f}_{\omega\omega} \neq 0$	$\partial_t + u\partial_u, \partial_x$	$A_{2.1}$
2	$e^{t+x}\tilde{f}(\omega)$, $\omega = ue^{-t-x}, \tilde{f}_{\omega\omega} \neq 0$	$\partial_t + u\partial_u$, $\partial_x + u\partial_u$	$A_{2.1}$
3	$(t-x)^{-3}\tilde{f}(\omega)$, $\omega = (t-x)u, \tilde{f}_{\omega\omega} \neq 0$	$-t\partial_t - x\partial_x + u\partial_u$, $\partial_t + \partial_x$	$A_{2.2}$
4	$x^{-1}\tilde{f}(\omega)$, $\omega = x^{-1}u, \tilde{f}_{\omega\omega} \neq 0$	$-t\partial_t - x\partial_x - u\partial_u$, ∂_t	$A_{2.2}$
5	$(t-x)^{-2}\tilde{f}(u)$, $\tilde{f}_{uu} \neq 0$	$\partial_t + \partial_x$, $t\partial_t + x\partial_x$, $t^2\partial_t + x^2\partial_x$	$sl(2,R)$
6	$\exp(x^{-1}u)$	$-t\partial_t + x\partial_u$, $\partial_t, x\partial_x + u\partial_u$	$A_{2.2} \oplus A_1$
7	$\lambda\|x\|^{-m-2}\|u\|^{m+1}$, $\lambda \neq 0, m \neq 0, -, 1-2$	$\partial_t, t\partial_t - \frac{1}{m}u\partial_u$, $x\partial_x + \frac{m+1}{m}u\partial_u$	$A_{2.2} \oplus A_1$
8	$\tilde{f}(u), \tilde{f}_{uu} \neq 0$	$\partial_t, \partial_x, -t\partial_t - x\partial_x$	$A_{3.6}$
9	$\lambda\|u\|^{n+1}, \lambda \neq 0, n \neq 0, -1$	$t\partial_t - \frac{1}{n}u\partial_u$, $x\partial_x - \frac{1}{n}u\partial_u$, ∂_t, ∂_x	$A_{2.2} \oplus A_{2.2}$

6 Group classification of equation (2.7)

The first step of the algorithm of group classification of (2.7)

$$u_{tt} = u_{xx} + F(t, x, u, u_x), \quad F_{u_x u_x} \neq 0$$

has been partially performed in the second chapter. It follows from Theorem 1 that the invariance group of equation (2.7) is generated by infinitesimal operator (2.4). What is more, the real constants $\lambda, \lambda_1, \lambda_2$ and real-valued functions $h = h(x), r = r(t,x)$, $F = F(t,x,u,u_x)$ obey the relation (2.5). The equivalence group of the class of equations (2.7) is formed by transformations (3.1).

The above enumerated facts enable using results of group classification of equation (2.8) in order to classify equation (2.7). In particular, using Lemmas 1 and 2 it is straightforward to verify that the following assertions hold true.

Theorem 11 *There are, at most, seven inequivalent classes of nonlinear equations (2.7) invariant under the one-dimensional Lie algebras.*

Below we give the full list of the invariant equations and the corresponding invariance algebras.

$$
\begin{aligned}
A_1^1 &= \langle t\partial_t + x\partial_x\rangle : \quad F = t^{-2}G(\xi, u, \omega),\ \xi = tx^{-1},\ \omega = xu_x;\\
A_1^2 &= \langle \partial_t + k\partial_x\rangle\ (k>0) : \quad F = G(\eta, u, u_x),\ \eta = x - kt;\\
A_1^3 &= \langle \partial_x\rangle : \quad F = G(t, u, u_x);\\
A_1^4 &= \langle \partial_t\rangle : \quad F = G(x, u, u_x);\\
A_1^5 &= \langle \partial_t + f(x)u\partial_u\rangle\ (f\neq 0) :\\
&\qquad F = -tf''u + t^2(f')^2u - 2tf'u_x + e^{tf}G(x, v, \omega),\\
&\qquad v = e^{-tf}u,\quad \omega = u^{-1}u_x - f'f^{-1}\ln|u|;\\
A_1^6 &= \langle f(x)u\partial_u\rangle\ (f\neq 0) : \quad F = -f^{-1}f''u\ln|u|\\
&\qquad -2f^{-1}f'u_x\ln|u| + f^{-2}(f')^2u\ln^2|u| + uG(t,x,\omega),\\
&\qquad \omega = u^{-1}u_x - f'f^{-1}\ln|u|;\\
A_1^7 &= \langle f(t,x)\partial_u\rangle\ (f\neq 0) : \quad F = f^{-1}(f_{tt} - f_{xx})u + G(t,x,\omega),\\
&\qquad \omega = u_x - f^{-1}f_xu.
\end{aligned}
$$

Note that if the functions F and G are arbitrary, then the given algebras are maximal (in Lie's sense) symmetry algebras of the respective equations.

Theorem 12 *An equation of the form (2.7) cannot admit Lie algebra which has a subalgebra having nontrivial Levi factor.*

With account of the above facts we conclude that nonlinear equations (2.7) admit a symmetry algebra of the dimension higher than one only if the latter is a solvable real Lie algebra. That is why, we turn to classifying equations (2.7) whose invariance algebras are two-dimensional solvable Lie algebras.

As the calculations are similar to those performed in the third section, we present the final result only. Namely, we give the form of invariant equations and the corresponding realizations of the two-dimensional invariance algebras.

I. $A_{2.1}$-invariant equations

$$
\begin{aligned}
A_{2.1}^1 &= \langle t\partial_t + x\partial_x,\ u\partial_u\rangle : \quad F = x^{-2}uG(\xi,\omega),\\
&\qquad \xi = tx^{-1},\ \omega = u^{-1}xu_x;\\
A_{2.1}^2 &= \langle t\partial_t + x\partial_x,\ \sigma(\xi)\partial_u\rangle\ (\sigma\neq 0,\ \xi = tx^{-1}) :\\
&\qquad F = x^{-2}[\sigma^{-1}((1-\xi^2)\sigma'' - 2\xi\sigma')u + G(\xi,\omega)],\\
&\qquad \omega = \xi\sigma'u + \sigma xu_x;
\end{aligned}
$$

$$
\begin{aligned}
A^3_{2.1} &= \langle \partial_t + k\partial_x,\ u\partial_u\rangle\ (k>0): \quad F = uG(\eta,\omega),\\
&\quad \eta = x - kt,\ \omega = u^{-1}u_x;\\
A^4_{2.1} &= \langle \partial_t + k\partial_x,\ \varphi(\eta)\partial_u\rangle\ (k>0,\ \eta = x-kt,\ \varphi\neq 0):\\
&\quad F = (k^2-1)\varphi''\varphi^{-1}u + G(\eta,\omega),\ \omega = \varphi u_x - \varphi' u;\\
A^5_{2.1} &= \langle \partial_t + k\partial_x,\ \partial_x + u\partial_u\rangle\ (k>0):\\
&\quad F = e^{\eta}\, G(\omega, v),\ \eta = x-kt,\ \omega = ue^{-\eta},\ v = u^{-1}u_x;\\
A^6_{2.1} &= \langle \partial_t,\ \partial_x\rangle: \quad F = G(u,u_x);\\
A^7_{2.1} &= \langle \partial_x,\ u\partial_u\rangle: \quad F = uG(t,\omega),\ \omega = u^{-1}u_x;\\
A^8_{2.1} &= \langle \partial_x,\ \varphi(t)\partial_u\rangle\ (\varphi\neq 0):\\
&\quad F = \varphi^{-1}\varphi'' u + G(t,u_x);\\
A^9_{2.1} &= \langle \partial_t,\ \partial_u\rangle: \quad F = G(x,u_x);\\
A^{10}_{2.1} &= \langle \partial_t,\ f(x)u\partial_u\rangle\ (f\neq 0):\\
&\quad F = -u^{-1}u_x^2 + uG(x,\omega);\\
&\quad \omega = u^{-1}u_x - f'f^{-1}\ln|u|;\\
A^{11}_{2.1} &= \langle \partial_t + f(x)u\partial_u,\ g(x)u\partial_u\rangle\ (\delta = f^{-1}f' - g^{-1}g'\neq 0):\\
&\quad F = -g^{-1}g''u\ln|u| - 2g^{-1}g'u_x\ln|u|\\
&\qquad + g^{-2}(g')^2u\ln^2|u| - 2f\delta t u_x + 2f\delta g' g^{-1} t u\ln|u|\\
&\qquad + f^2\delta^2 t^2 u + f(g^{-1}g'' - f^{-1}f'')tu + uG(x,\omega),\\
&\quad \omega = u^{-1}u_x - g'g^{-1}\ln|u| - tf\delta;\\
A^{12}_{2.1} &= \langle \partial_t + f(x)u\partial_u,\ e^{tf}\partial_u\rangle\ (f\neq 0):\\
&\quad F = [f^2 - tf'' + t^2(f')^2]u - 2tf'u_x + e^{tf}G(x,\omega),\\
&\quad \omega = e^{-tf}(u_x - tf'u);\\
A^{13}_{2.1} &= \langle f(x)u\partial_u,\ g(x)u\partial_u\rangle\ (\delta = f'g - g'f\neq 0):\\
&\quad F = -u^{-1}u_x^2 - \delta^{-1}\delta' u_x\\
&\qquad + \delta^{-1}[f''g' - g''f']u\ln|u| + uG(t,x);\\
A^{14}_{2.1} &= \langle \varphi(t)\partial_u,\ \psi(t)\partial_u\rangle\ (\varphi'\psi - \varphi\psi'\neq 0):\\
&\quad F = \varphi^{-1}\varphi'' u + G(t,x,u_x),\ \varphi''\psi - \varphi\psi'' = 0.
\end{aligned}
$$

II. $A_{2.2}$-invariant equations

$$
\begin{aligned}
A^1_{2.2} &= \langle t\partial_t + x\partial_x,\ xu\partial_u\rangle: \quad F = x^{-2}u\ln^2|u|\\
&\qquad -2x^{-1}u_x\ln|u| + t^{-2}uG(\xi,\omega),\ \xi = tx^{-1};\\
&\quad \omega = xu^{-1}u_x - \ln|u|;\\
A^2_{2.2} &= \langle t\partial_t + x\partial_x,\ t\varphi(\xi)\partial_u\rangle\ (\varphi\neq 0,\ \xi = tx^{-1}):\\
&\quad F = t^{-2}(1-\xi^2)\varphi^{-1}\xi(2\varphi' + \xi\varphi'')u + t^{-2}G(\xi,\omega),\\
&\quad \omega = x\varphi u_x + \xi\varphi' u;
\end{aligned}
$$

$$\begin{aligned}
A^3_{2.2} &= \langle \partial_t + k\partial_x,\ \exp(k^{-1}x)u\partial_u\rangle\ (k>0):\\
&\quad F = k^{-2}u\ln^2|u| - 2k^{-1}u_x\ln|u| - k^{-2}u\ln|u|\\
&\quad +uG(\eta,\omega),\ \eta = x-kt,\ \omega = u^{-1}u_x - k^{-1}\ln|u|;\\
A^4_{2.2} &= \langle \partial_t + k\partial_x,\ e^t\varphi(\eta)\partial_u\rangle\ (\eta = x-kt,\ k>0,\ \varphi\neq 0):\\
&\quad F = \left((k^2-1)\varphi''\varphi^{-1} - 2k\varphi'\varphi^{-1} + 1\right)u + G(\eta,\omega),\\
&\quad \omega = \varphi u_x - \varphi' u,\ \varphi' = \frac{d\varphi}{d\eta};\\
A^5_{2.2} &= \langle -t\partial_t - x\partial_x,\ \partial_t + k\partial_x\rangle\ (k>0):\\
&\quad F = \eta^{-2}G(u,\omega),\ \eta = x-kt,\quad \omega = u_x\eta;\\
A^6_{2.2} &= \langle -t\partial_t - x\partial_x + mu\partial_u,\ \partial_t + k\partial_x\rangle\ (k>0,\ m\neq 0):\\
&\quad F = |\eta|^{-2-m}G(v,\omega),\ \eta = x-kt,\\
&\quad \omega = u|\eta|^m,\ v = u_x|\eta|^{m+1};\\
A^7_{2.2} &= \langle \partial_x,\ e^x u\partial_u\rangle:\quad F = u\ln^2|u| - u\ln|u| - 2u_x\ln|u|\\
&\quad +uG(t,\omega),\ \omega = u^{-1}u_x - \ln|u|;\\
A^8_{2.2} &= \langle \partial_x,\ e^x\varphi(t)\partial_u\rangle\ (\varphi\neq 0):\\
&\quad F = (\varphi^{-1}\varphi''-1)u + G(t,\omega),\ \omega = u_x - u;\\
A^9_{2.2} &= \langle -t\partial_t - x\partial_x,\ \partial_x\rangle:\quad F = t^{-2}G(u,tu_x);\\
A^{10}_{2.2} &= \langle -t\partial_t - x\partial_x + ku\partial_u,\ \partial_x\rangle,\ (k\neq 0):\\
&\quad F = |t|^{-2-k}G(v,\omega),\ v = |t|^k u,\ \omega = |t|^{k+1}u_x;\\
A^{11}_{2.2} &= \langle \partial_t,\ e^t\partial_u\rangle:\quad F = u + G(x,u_x);\\
A^{12}_{2.2} &= \langle -t\partial_t - x\partial_x,\ \partial_t\rangle:\quad F = x^{-2}G(u,\omega),\ \omega = xu_x;\\
A^{13}_{2.2} &= \langle \partial_t + f(x)u\partial_u,\ e^{(1+f)t}\partial_u\rangle\ (f\neq 0):\\
&\quad F = -\left(tf'' - t^2(f')^2 - (1+f^2)\right)u - 2tf'u_x\\
&\quad +e^{tf}G(x,\omega),\ \omega = e^{-tf}\left(u_x - f'(t+f^{-1})u\right);\\
A^{14}_{2.2} &= \langle -t\partial_t - x\partial_x,\ \partial_t + kx^{-1}u\partial_u\rangle\ (k>0);\\
&\quad F = -2ktx^{-3}u + k^2t^2x^{-4}u + 2ktx^{-2}u_x\\
&\quad +x^{-2}\exp(ktx^{-1})G(v,\omega), v = \exp(-kx^{-1}t)u,\\
&\quad \omega = xu^{-1}u_x + \ln|u|;\\
A^{15}_{2.2} &= \langle k(t\partial_t + x\partial_x),\ |x|^{k^{-1}}u\partial_u\rangle\ (k\neq 0,1):\\
&\quad F = -k^{-2}(1-k)x^{-2}u\ln|u| - 2k^{-1}x^{-1}u_x\ln|u|\\
&\quad\quad +k^{-2}x^{-2}u\ln^2|u| + x^{-2}uG(v,\omega),\\
&\quad v = tx^{-1},\ \omega = xu^{-1}u_x - k^{-1}\ln|u|;\\
A^{16}_{2.2} &= \langle k(t\partial_t + x\partial_x),\ |t|^{k^{-1}}\varphi(\xi)\partial_u\rangle\ (k\neq 0,1,\ \varphi\neq 0,\\
&\quad \xi = tx^{-1}) : F = [k^{-1}(k^{-1}-1) + 2\xi(k^{-1}-\xi^2)\varphi^{-1}\varphi'\\
&\quad\quad +\xi^2(1-\xi)^2\varphi^{-1}\varphi'']t^{-2}u + t^{-2}G(\xi,\omega),\\
&\quad \omega = x\varphi u_x + \xi\varphi' u.
\end{aligned}$$

In the above formulas G stands for an arbitrary smooth function. As usual, prime denotes the derivative of a function of one variable.

6.1 Group classification of equation $u_{tt} = u_{xx} - u^{-1}u_x^2 + A(x)u_x + B(x)u\ln|u| + uD(t,x)$

Before analyzing equations (2.7) admitting algebras of the dimension higher than two we perform group classification of the equation

$$u_{tt} = u_{xx} - u^{-1}u_x^2 + A(x)u_x + B(x)u\ln|u| + uD(t,x). \tag{6.1}$$

Here $A(x), B(x), D(t,x)$ are arbitrary smooth functions. Note that the above class of PDEs contains $A_{2.1}^{13}$-invariant equation. Importantly, class (6.1) contains a major part of equations of the form (2.7), whose maximal symmetry algebras have dimension three or four. This fact is used to simplify group classification of equations (2.7).

Lemma 5 *If A, B and D are arbitrary, then the maximal invariance algebra of PDE (6.1) is the two-dimensional Lie algebra equivalent to $A_{2.1}^{13}$ and (6.1) reduces to $A_{2.1}^{13}$-invariant equation. Next, if the maximal symmetry algebra of an equation of the form (6.1) is three-dimensional (we denote it as A_3), then this equation is equivalent to one of the following ones:*

I. $A_3 \sim A_{3.1},\ A_3 = \langle \partial_t, f(x)u\partial_u, \varphi(x)u\partial_u\rangle,$
$A = -\sigma^{-1}\sigma',\ B = \sigma^{-1}\rho,\ D = 0, \sigma = f'\varphi - f\varphi' \neq 0,$
$\rho = \varphi' f'' - \varphi'' f';$

II. $A_3 \sim A_{3.1},\ A_3 = \langle f(x)u\partial_u, \varphi(x)u\partial_u, \partial_t + \psi(x)u\partial_u\rangle,$
$A = -\sigma^{-1}\sigma',\ B = \sigma^{-1}\rho,$
$D = t\sigma^{-1}[\sigma'\psi' - \psi\rho - \sigma\psi''],$
$\sigma = f'\varphi - \varphi' f \neq 0, \rho = f''\varphi' - \varphi'' f',$
$f'\psi - f\psi' \neq 0, \quad \varphi'\psi - \varphi\psi' \neq 0;$

III. $D = x^{-2}G(\xi),\ \xi = tx^{-1},\ G \neq 0:$

1) $A_3 \sim A_{3.2},\ A_3 = \langle t\partial_t + x\partial_x, u\partial_u, |x|^{1-n}u\partial_u\rangle,$
$A = nx^{-1}\ (n \neq 1),\ B = 0;$

2) $A_3 \sim A_{3.3},\ A_3 = \langle t\partial_t + x\partial_x, u\partial_u, u\ln|x|\partial_u\rangle,\ A = x^{-1},\ B = 0;$

3) $A_3 \sim A_{3.4}, A_3 = \langle t\partial_t + x\partial_x, \sqrt{|x|}u\partial_u, \sqrt{|x|}\ln|x|u\partial_u\rangle,$
$A = 0,\ \ B = \frac{1}{4}x^{-2};$

4) $A_3 \sim A_{3.9}, A_3 = \langle t\partial_t + x\partial_x, \sqrt{|x|}\cos(\frac{1}{2}\beta\ln|x|)u\partial_u,$
$\sqrt{|x|}\sin(\frac{1}{2}\beta\ln|x|)u\partial_u\rangle,\ A = 0, B = mx^{-2},$
$m > \frac{1}{4},\ \beta = \sqrt{4m-1};$

5) $A_3 \sim A_{3.7}, A_3 = \langle t\partial_t + x\partial_x, (\sqrt{|x|})^{1+\beta}u\partial_u, (\sqrt{|x|})^{1-\beta}u\partial_u\rangle,$
$A = 0, B = mx^{-2},\ m < \frac{1}{4},\ m \neq 0,\ \beta = \sqrt{1-4m};$

6) $A_3 \sim A_{3.8}, A_3 = \langle t\partial_t + x\partial_x, \cos(\sqrt{m}\ln|x|)u\partial_u,$
$\sin(\sqrt{m}\ln|x|)u\partial_u\rangle,\ A = x^{-1}, B = mx^{-2},\ m > 0;$

7) $A_3 \sim A_{3.6}, A_3 = \langle t\partial_t + x\partial_x, |x|^{\sqrt{|m|}}u\partial_u, |x|^{-\sqrt{|m|}}u\partial_u\rangle,$
$A = x^{-1},\ B = mx^{-2},\ m < 0;$

8) $A_3 \sim A_{3.4}, A_3 = \langle t\partial_t + x\partial_x, (\sqrt{|x|})^{1-n} u\partial_u, (\sqrt{|x|})^{1-n}$
$\times \ln|x| u\partial_u\rangle,\ A = nx^{-1}\ (n \neq 0, 1),\ B = \frac{1}{4}(n-1)^2 x^{-2};$

9) $A_3 \sim A_{3.9}, A_3 = \langle t\partial_t + x\partial_x, (\sqrt{|x|})^{1-n} \cos(\frac{1}{2}\beta \ln|x|) u\partial_u,$
$(\sqrt{|x|})^{1-n} \sin(\frac{1}{2}\beta \ln|x|) u\partial_u\rangle,\ A = nx^{-1}\ (n \neq 0, 1),$
$B = mx^{-2}\ (m > \frac{1}{4}(n-1)^2),\ \beta = \sqrt{4m - (n-1)^2};$

10) $A_3 \sim A_{3.7}, A_3 = \langle t\partial_t + x\partial_x, (\sqrt{|x|})^{1-\beta-n} u\partial_u, (\sqrt{|x|})^{1-n+\beta}$
$\times u\partial_u\rangle,\ A = nx^{-1}\ (n \neq 0, 1),\ B = mx^{-2}$
$(m < \frac{1}{4}(n-1)^2,\ m \neq 0),\ \beta = \sqrt{(n-1)^2 - 4m}.$

IV. $D = G(t)$,

1) $A_3 \sim A_{3.3},\ A_3 = \langle \partial_x, u\partial_u, xu\partial_u\rangle,$
$A = B = 0;$

2) $A_3 = A_{3.2},\ A_3 = \langle \partial_x, u\partial_u, e^x u\partial_u\rangle,$
$A = -1,\ B = 0;$

3) $A_3 \sim A_{3.8},\ A_3 = \langle \partial_x, \cos(x) u\partial_u, \sin(x) u\partial_u\rangle,$
$A = 0,\ B = 1;$

4) $A_3 \sim A_{3.6},\ A_3 = \langle \partial_x, e^x u\partial_u, e^{-x} u\partial_u\rangle,$
$A = 0,\ B = -1;$

5) $A_3 \sim A_{3.4},\ A_3 = \langle \partial_x, \exp\left(\frac{1}{2}x\right) u\partial_u, \exp\left(\frac{1}{2}x\right) xu\partial_u\rangle,$
$A = -1,\ B = \frac{1}{4};$

6) $A_3 \sim A_{3.7},\ A_3 = \langle \partial_x, \exp\left(\frac{1}{2}(1+\beta)x\right) u\partial_u, \exp\left(\frac{1}{2}(1-\beta)x\right) u\partial_u\rangle,$
$A = -1, B = m\ (m < \frac{1}{4}),\ m \neq 0, \beta = \sqrt{1-4m};$

7) $A_3 \sim A_{3.9},\ A_3 = \langle \partial_x, \exp\left(\frac{1}{2}x\right) \cos(\frac{1}{2}\beta x) u\partial_u, \exp\left(\frac{1}{2}x\right) \sin(\frac{1}{2}\beta x) u\partial_u\rangle,$
$A = -1, B = m\ (m > \frac{1}{4}),\ \beta = \sqrt{4m-1};$

V. $D = G(\eta),\ \eta = x - kt,\ k > 0,$

1) $A_3 \sim A_{3.3},\ A_3 = \langle \partial_t + k\partial_x, u\partial_u, xu\partial_u\rangle,$
$A = B = 0;$

2) $A_3 = A_{3.2},\ A_3 = \langle \partial_t + k\partial_x, u\partial_u, e^x u\partial_u\rangle,$
$A = -1,\ B = 0;$

3) $A_3 \sim A_{3.8},\ A_3 = \langle \partial_t + k\partial_x, \cos(x) u\partial_u, \sin(x) u\partial_u\rangle,$
$A = 0,\ B = 1;$

4) $A_3 \sim A_{3.6},\ A_3 = \langle \partial_t + k\partial_x, e^x u\partial_u, e^{-x} u\partial_u\rangle,$
$A = n,\ B = -1;$

5) $A_3 \sim A_{3.4},\ A_3 = \langle \partial_t + k\partial_x, \exp\left(\frac{1}{2}x\right) u\partial_u, \exp\left(\frac{1}{2}x\right) xu\partial_u\rangle,$
$A = -1,\ B = \frac{1}{4};$

6) $A_3 \sim A_{3.7},\ A_3 = \langle \partial_t + k\partial_x, \exp\left(\frac{1}{2}(1+\beta)x\right) u\partial_u, \exp\left(\frac{1}{2}(1-\beta)x\right) u\partial_u \rangle,$

$A = -1, B = m\ (m < \frac{1}{4}),\ m \neq 0, \beta = \sqrt{1-4m};$

7) $A_3 \sim A_{3.9},\ A_3 = \langle \partial_t + k\partial_x, \exp\left(\frac{1}{2}x\right) \cos(\frac{1}{2}\beta x) u\partial_u,$

$\exp\left(\frac{1}{2}x\right) \sin(\frac{1}{2}\beta x) u\partial_u \rangle,\ A = -1,\ B = m\ (m > \frac{1}{4})\ \beta = \sqrt{4m-1}.$

Proof. Inserting the expression

$$F = -u^{-1}u_x^2 + A(x)u_x + B(x)u \ln|u| + uD(t,x)$$

into classifying equation (2.5) we get the system of determining equations for the functions $h(x)$, $r(t,x)$ and constants $\lambda, \lambda_1, \lambda_2$:

$$\begin{aligned} &r = 0, \quad (\lambda x + \lambda_2)A' + \lambda A = 0, \\ &(\lambda x + \lambda_2)B' + 2\lambda B = 0, \quad h'' + Ah' + Bh = -(\lambda t + \lambda_1)D_t - (\lambda x + \lambda_2)D_x - 2\lambda D. \end{aligned} \tag{6.2}$$

First, consider the case of arbitrary functions A, B, D. The left-hand side of the fourth equation of (6.2) depends on x only. What is more, since D is arbitrary, relation $D_t \not\equiv 0$ holds. Hence it immediately follows that the constants $\lambda, \lambda_1, \lambda_2$ must be equal to zero. As a consequence, the fourth equation becomes linear ordinary differential equation for the function $h(x)$

$$h'' + Ah' + Bh = 0. \tag{6.3}$$

The general solution of the above equation reads as

$$h = C_1 f(x) + C_2 \varphi(x), \quad C_1, C_2 \in \mathbb{R},$$

$f(x)$ and $\varphi(x)$ being the fundamental system of solutions of the equation

$$y'' + Ay' + By = 0, \quad y = y(x). \tag{6.4}$$

Inserting this expression into (6.3) yields

$$A = -\sigma^{-1}\sigma',\ B = \sigma^{-1}(\varphi' f'' - f'\varphi''),$$

where $\sigma = \varphi f' - \varphi' f \neq 0$, which proves the first part of lemma.

Suppose now that $D = 0$. Then, if at least one of the functions A or B is arbitrary, then $\lambda = \lambda_2 = 0$ and the function h is a solution of (6.3). This completes the proof of the case I of the second part of the lemma statement.

Provided functions A and B are not arbitrary, it follows from the second and third equations of (6.2) that one of the following relations

$$\begin{aligned} &1) \quad A = B = 0; \\ &2) \quad A = n,\ B = m,\ m, n \in \mathbb{R},\ |n| + |m| \neq 0; \\ &3) \quad A = nx^{-1},\ B = mx^{-2},\ m, n \in \mathbb{R},\ |n| + |m| \neq 0 \end{aligned} \tag{6.5}$$

holds. With these conditions the maximal invariance algebra of (6.1) has the dimension higher than three. Consequently, without loss of generality we can suggest that $D \neq 0$. Integrating the equation

$$(\lambda t + \lambda_1)D_t + (\lambda x + \lambda_2)D_x + 2\lambda D = H(x),$$

under $D \neq 0$ yields the following (inequivalent) expressions for the function $D(t,x)$:

$$\begin{aligned} D &= x^{-2}G(\xi) + x^{-2}\int xH(x)\,dx,\ \ \xi = tx^{-1}; \\ D &= G(\eta) + k^{-1}\int H(x)\,dx, \eta = x - kt,\ \ k > 0; \\ D &= G(t) + \int H(x)\,dx, \\ D &= tH(x) + \tilde{H}(x). \end{aligned} \tag{6.6}$$

The change of variables

$$\bar{t} = t, \quad \bar{x} = x, \quad u = \theta(x)v(\bar{t},\bar{x}), \quad \theta \neq 0, \tag{6.7}$$

where θ is a solution of equation

$$\theta^{-1}\theta'' - \theta^{-2}(\theta')^2 + A\theta^{-1}\theta' + B\ln|\theta| + \Lambda(x) = 0,$$

preserves the form of equation (6.1). We can use this fact to simplify the form of the function D. As a result, we get

$$\begin{aligned} D &= x^{-2}G(\xi),\ \ \xi = tx^{-1}; \\ D &= G(\eta),\ \ \eta = x - kt,\ \ k > 0; \\ D &= G(t), \\ D &= tH(x). \end{aligned} \tag{6.8}$$

If the function D is given by the one of the first three expressions, then $H(x) \equiv 0$ and h satisfies (6.3).

Given the condition $D = tH(x)$, we have

$$h'' + Ah' + Bh = -\lambda_1 H, \quad (\lambda x + \lambda_2)H' + 3\lambda H = 0.$$

So that the maximal invariance algebra of the corresponding equation (6.1) is three-dimensional iff $\lambda = \lambda_2 = 0$, which yields the case II of the second part of the lemma statement.

Turn now to the case when $D = x^{-2}G(\xi),\ \xi = tx^{-1}$. Then the function $G \neq 0$ obey the equation

$$(\lambda_2\xi - \lambda_1)G' + 2\lambda_2 G = 0. \tag{6.9}$$

If G is an arbitrary function, then $\lambda_1 = \lambda_2 = 0$. In addition, we have $\lambda \neq 0$ (otherwise the maximal invariance algebra is two-dimensional). Hence we get

$$xA' + a = 0, \quad xB' + 2B = 0.$$

Consequently, functions A and B are given by either first or third formula from (6.5). Analyzing these expressions yields ten cases of the case III of the second part of the lemma statement.

If the function G is not arbitrary, then integrating (6.9) we get

$$\begin{aligned} &G = p, \quad p \in \mathbb{R},\ p \neq 0; \\ &G = p(\xi - q)^{-2},\ \ p \neq 0,\ \ q \geq 0. \end{aligned}$$

Given the condition $G = p$, the parameter λ_2 vanishes. Hence in view of the requirement for the maximal algebra to be three-dimensional, it follows that λ vanishes as well. This yields the case when A and B in (6.1) are arbitrary functions (the case I of the second part of the lemma statement). If $G = p(\xi - q)^{-2},\ p \neq 0$, then $\lambda_1 = \lambda_2 q$. Hence we conclude that the maximal invariance algebra of the corresponding equation (6.1) is three-dimensional iff the functions A, B are given by formulas 3) from (6.5) (which implies that $\lambda_1 = \lambda_2 = 0$) and we get the case III of the second part of the lemma

statement. Next, if A, B are given by formulas 2 from (6.5) (which implies that $\lambda = 0,\ D = p(t-qx)^{-2}$) and we arrive at the case IV ($q = 0$) or the case V ($q > 0$) of the second part of the lemma statement.

Turn now to the case $D = G(\eta),\ \eta = x - kt,\ k > 0$. If these relations hold, then

$$\lambda(\eta G' + 2G) + (\lambda_2 - k\lambda_1)G' = 0.$$

Hence it follows that if G is an arbitrary function of η, then $\lambda = 0,\ \lambda_2 = k\lambda_1$. That is why, the maximal invariance algebra of (6.1) is three-dimensional iff either $A = B = 0$ or A, B are given by formulas 2 from (6.5). So that we have obtained all equations listed in the case V of the second part of the lemma statement.

The cases when either $G = p$ ($p \neq 0$) or $G = p\eta^{-2}$ ($p \neq 0$) yield no new invariant equations (6.5).

Consider now the last possible case $D = G(t)$. If this is the case, then the equation

$$(\lambda t + \lambda_1)G' + 2\lambda G = 0$$

holds. Hence if follows that if G is an arbitrary function, then $\lambda = \lambda_1 = 0$. So that the maximal invariance algebra of equation (6.1) is three-dimensional iff A, B are given by the formula 2 from (6.5) and we have obtained all the invariant equations from the case IV of the second part of assertion of the lemma. If either of relations $G = p$ ($p \neq 0$) or $G = pt^{-2}$ ($p \neq 0$) hold, them no new invariant equations having three-dimensional maximal invariance algebras can be obtained.

To complete the proof of the lemma, we need to establish non equivalence of the obtained invariant equations. To this end it suffices to prove that there are no transformations from the group $\mathcal{E}$, reducing their invariance algebras one into another.

As we already mentioned in Section 3 there exist nine non-isomorphic three-dimensional solvable Lie algebras $A_{3.i} = \langle e_1, e_2, e_3\rangle$ ($i = 1, 2, \ldots, 9$). We analyze in some detail the case of the algebra $A_{3.3}$. The list of invariant equations and algebras contains three algebras which are isomorphic to $A_{3.3}$, namely,

$$\begin{aligned}
L_1 &= \langle t\partial_t + x\partial_x, u\partial_u, u\ln|x|\partial_u\rangle;\\
L_2 &= \langle \partial_x, u\partial_u, xu\partial_u\rangle;\\
L_2 &= \langle \partial_t + k\partial_x, u\partial_u, xu\partial_u\rangle\ (k > 0).
\end{aligned}$$

Denote the basis elements of the algebra L_2 as e_1, e_2, e_3. Suppose that there is a transformation φ from the group $\mathcal{E}$ transforming L_2 into L_3. In other words we suppose that there exist constants $\alpha_i, \beta_i, \delta_i \in R$ ($i = 1, 2, 3$) such that the relations

$$\varphi(e_1) = \sum_{i=1}^{3}\alpha_i\tilde{e}_i,\quad \varphi(e_2) = \sum_{i=1}^{3}\beta_i\tilde{e}_i,\quad \varphi(e_3) = \sum_{i=1}^{3}\delta_i\tilde{e}_i$$

and

$$\triangle = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \delta_1 & \delta_2 & \delta_3 \end{vmatrix} \neq 0$$

hold. In the above formulas $\tilde{e}_1 = \partial_{\overline{t}} + k\partial_{\overline{x}},\ \tilde{e}_2 = v\partial_v,\ \tilde{e}_3 = \overline{x}v\partial_v$. Equating the coefficients of the linearly independent operators $\partial_{\overline{t}}, \partial_{\overline{x}}, \partial_v$ yields that $\alpha_1 = \beta_1 = \delta_1 = 0$. Hence we get the contradictory equation $\triangle = 0$. This means that realizations L_2 and L_3 are non-isomorphic. Analogously, we prove that L_1 and L_2 (as well as L_1 and L_3) are non isomorphic.

The remaining algebras are considered in a similar way. The Lemma is proved.

In what follows we will use the results on classification of abstract four-dimensional solvable real Lie algebras $A_4 = \langle e_1, e_2, e_3, e_4\rangle$ [43, 44]. There are ten decomposable

$$\begin{aligned}
&4A_1 = 3A_1 \oplus A_1 = A_{3.1} \oplus A_1,\quad A_{2.2} \oplus 2A_1 = A_{2.2} \oplus A_{2.1} = A_{3.2} \oplus A_1,\\
&2A_{2.2} = A_{2.2} \oplus A_{2.2},\quad A_{3.i} \oplus A_1\ (i = 3, 4, \ldots, 9);
\end{aligned}$$

and ten non-decomposable four-dimensional solvable real Lie algebras (note that we give below non-zero commutation relations only).

$$\begin{aligned}
A_{4.1} &: [e_2, e_4] = e_1, \quad [e_3, e_4] = e_2; \\
A_{4.2} &: [e_1, e_4] = qe_1, \quad [e_2, e_4] = e_2, \\
& \quad [e_3, e_4] = e_2 + e_3, \quad q \neq 0; \\
A_{4.3} &: [e_1, e_4] = e_1, \quad [e_3, e_4] = e_2; \\
A_{4.4} &: [e_1, e_4] = e_1, \quad [e_2, e_4] = e_1 + e_2, \\
& \quad [e_3, e_4] = e_2 + e_3; \\
A_{4.5} &: [e_1, e_4] = e_1, \quad [e_2, e_4] = qe_2, \\
& \quad]e_3, e_4] = pe_3, \quad -1 \leq p \leq q \leq 1, \quad p \cdot q \neq 0; \\
A_{4.6} &: [e_1, e_4] = qe_1, \quad [e_2, e_4] = pe_2 - e_3, \\
& \quad [e_3, e_4] = e_2 + pe_3, \quad q \neq 0, \quad p \geq 0; \\
A_{4.7} &: [e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \\
& \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3; \\
A_{4.8} &: [e_2, e_3] = e_1, \quad [e_1, e_4] = (1 + q)e_1, \\
& \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = qe_3, \quad |q| \leq 1; \\
A_{4.9} &: [e_2, e_3] = e_1, \quad [e_1, e_4] = 2qe_1, \\
& \quad [e_2, e_4] = qe_2 - e_3, \quad [e_3, e_4] = e_2 + qe_3, \quad q \geq 0; \\
A_{4.10} &: [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \\
& \quad [e_1, e_4] = -e_2, \quad [e_2, e_4] = e_1.
\end{aligned}$$

Theorem 13 *Equation $u_{tt} = u_{xx} - u^{-1}u_x^2$ has the widest symmetry group amongst equations of the form (6.1). Its maximal invariance algebra is the five-dimensional Lie algebra*

$$A_5^1 = \langle \partial_t, \partial_x, t\partial_t + x\partial_x, xu\partial_u, u\partial_u \rangle.$$

There are no equations of the form (6.1) which are inequivalent to the above equation and admit invariance algebra of the dimension higher than four. Inequivalent equations (6.1) admitting four-dimensional algebras are listed below together with their symmetry algebras.

I. $D = 0$,

1) $A_4 \sim A_{3.6} \oplus A_1, \quad A_4 = \langle \partial_t, \partial_x, u\,\mathrm{ch}(\beta x)\partial_u, u \sinh(\beta x)\partial_u \rangle,$
$A = 0, B = -\beta^2, \ \beta \neq 0;$

2) $A_4 \sim A_{3.8} \oplus A_1, \quad A_4 = \langle \partial_t, \partial_x, u \cos(\beta x)\partial_u, u \sin(\beta x)\partial_u \rangle,$
$A = 0, B = \beta^2, \ \beta \neq 0;$

3) $A_4 \sim A_{2.1} \oplus A_{2.2}, \ A_4 = \langle \partial_t, \partial_x, u\partial_u, e^{-x}u\partial_u \rangle, A = 1, B = 0;$

4) $A_4 \sim A_{3.4} \oplus A_1, \ A_4 = \langle \partial_t, \partial_x, e^{-x}u\partial_u, xe^{-x}u\partial_u \rangle, \ A = 2, B = 1;$

5) $A_4 \sim A_{3.9} \oplus A_1, \ A_4 = \langle \partial_t, \partial_x, ue^{-x} \cos(\beta x)\partial_u, ue^{-x} \sin(\beta x)\partial_u \rangle,$
$A = 2, B = m, m > 1, \ \beta = \sqrt{m-1};$

6) $A_4 \sim A_{3.7} \oplus A_1, \ A_4 = \langle \partial_t, \partial_x, ue^{-x}\mathrm{ch}(\beta x)\partial_u, ue^{-x} \sinh(\beta x)\partial_u \rangle,$
$A = 2, \ B = m, \ m > 1, \ m \neq 0, \beta = \sqrt{1-m};$

7) $A_4 \sim A_{4.2}$, $A_4 = \langle \partial_t, t\partial_t + x\partial_x, \sqrt{|x|}u\partial_u, u\sqrt{|x|}\ln|x|\partial_u\rangle$,
$A = 0$, $B = \frac{1}{4}x^{-2}$;

8) $A_4 \sim A_{4.5}$, $A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}+\beta}u\partial_u, |x|^{\frac{1}{2}-\beta}u\partial_u\rangle$, $A = 0$,
$B = mx^{-2}$, $m < \frac{1}{4}$, $m \neq 0$, $\beta = \sqrt{\frac{1}{4} - m}$;

9) $A_4 \sim A_{4.6}$, $A_4 = \langle \partial_t, t\partial_t + x\partial_x$,
$\sqrt{|x|}\cos(\beta\ln|x|)u\partial_u, \sqrt{|x|}\sin(\beta\ln|x|)u\partial_u\rangle$,
$A = 0$, $B = mx^{-2}$, $m > \frac{1}{4}$, $\beta = \sqrt{m - \frac{1}{4}}$;

10) $A_4 \sim A_{4.3}$, $A_4 = \langle \partial_t, t\partial_t + x\partial_x, u\ln|x|\partial_u, u\partial_u\rangle$, $A = x^{-1}$, $B = 0$;

11) $A_4 \sim A_{3.7} \oplus A_1$, $A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{1-n}u\partial_u, u\partial_u\rangle$,
$A = nx^{-1}$, $B = 0, n \neq 0, 1$;

12) $A_4 \sim A_{4.5}$, $A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}(1-n)}u\partial_u, |x|^{\frac{1}{2}(1-n)}u\ln|x|\partial_u\rangle$,
$A = nx^{-1}$, $B = \frac{1}{4}(n-1)^2x^{-2}$, $n \neq 0, 1$;

13) $A_4 \sim A_{4.5}$, $A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}(1-n+\beta)}u\partial_u, |x|^{\frac{1}{2}(1-n-\beta)}u\partial_u\rangle$,
$A = nx^{-1}$, $B = mx^{-2}$, $m < \frac{1}{4}(n-1)^2$, $m \neq 0$, $n \neq 0$,
$\beta = \sqrt{(n-1)^2 - 4m}$;

14) $A_4 \sim A_{4.6}$, $A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}(1-n)}\cos(\beta\ln|x|)u\partial_u$,
$|x|^{\frac{1}{2}(1-n)}\sin(\beta\ln|x|)u\partial_u\rangle$, $A = nx^{-1}$, $B = mx^{-2}$,
$m \neq 0$, $n \neq 0$, $m > \frac{1}{4}(n-1)^2$, $\beta = \sqrt{m - \frac{1}{4}(n-1)^2}$;

II. $D = ktx^{-3}$, $k > 0$,

1) $A_4 \sim A_{4.1}$, $A_4 = \langle \partial_t - \frac{1}{2}kx^{-1}u\partial_u, t\partial_t + x\partial_x, xu\partial_u, u\partial_u\rangle$, $A = B = 0$;

2) $A_4 \sim A_{4.2}$, $A_4 = \langle \partial_t - \frac{4}{9}kx^{-1}u\partial_u, t\partial_t + x\partial_x, \sqrt{|x|}u\partial_u, \sqrt{|x|}\ln|x|u\partial_u\rangle$,
$A = 0$, $B = \frac{1}{4}x^{-2}$;

3) $A_4 \sim A_{4.5}$, $A_4 = \langle \partial_t - \frac{k}{m+2}x^{-1}u\partial_u, t\partial_t + x\partial_x, |x|^{\frac{1}{2}+\beta}u\partial_u, |x|^{\frac{1}{2}-\beta}u\partial_u\rangle$,
$A = 0$, $B = mx^{-2}$, $m \neq 0, -2$, $m < \frac{1}{4}$, $\beta = \sqrt{\frac{1}{4} - m}$;

4) $A_4 \sim A_{4.2}$, $A_4 = \langle \partial_t + \frac{1}{9}kx^{-1}(1 + 3\ln|x|u)\partial_u, t\partial_t + x\partial_x$,
$x^2u\partial_u, x^{-1}u\partial_u\rangle$, $A = 0, B = -2x^{-2}$;

5) $A_4 \sim A_{4.6}$, $A_4 = \langle \partial_t - \frac{k}{m+2}x^{-1}u\partial_u, t\partial_t + x\partial_x, \sqrt{|x|}u\cos(\beta\ln|x|)\partial_u$,
$\sqrt{|x|}u\sin(\beta\ln|x|)\partial_u\rangle$, $A = 0, B = mx^{-2}$, $m > \frac{1}{4}, \beta = \sqrt{m - \frac{1}{4}}$;

6) $A_4 \sim A_{4.3}$, $A_4 = \langle \partial_t - kx^{-1}u\partial_u, t\partial_t + x\partial_x, u\partial_u, u\ln|x|\partial_u\rangle$,
$A = x^{-1}, B = 0$;

7) $A_4 \sim A_{3.4} \oplus A_1$, $A_4 = \langle \partial_t + kx^{-1}(1 + \ln|x|)u\partial_u, t\partial_t + x\partial_x, u\partial_u$,
$x^{-1}u\partial_u\rangle$, $A = 2x^{-1}$, $B = 0$;

8) $A_4 \sim A_{3.7} \oplus A_1$, $A_4 = \langle \partial_t + \frac{k}{n-2}x^{-1}u\partial_u, t\partial_t + x\partial_x, u\partial_u, |x|^{1-n}u\partial_u\rangle$,
$A = nx^{-1}, B = 0$, $n \neq 0, 1, 2$;

9) $A_4 = A_{4.4}$, $A_4 = \langle \partial_t - \frac{1}{2}kx^{-1}\ln^2|x|u\partial_u, t\partial_t + x\partial_x$,
$x^{-1}u\partial_u, x^{-1}\ln|x|u\partial_u\rangle$, $A = 3x^{-1}$, $B = x^{-2}$;

10) $A_4 \sim A_{4.2},\ A_4 = \langle \partial_t - \frac{4k}{(n-3)^2} x^{-1} u\partial_u, t\partial_t + x\partial_x, |x|^{\frac{1}{2}(1-n)} u\partial_u,$
$|x|^{\frac{1}{2}(1-n)} \ln |x| u\partial_u \rangle, \quad A = nx^{-1},\ B = \frac{1}{4}(n-1)^2 x^{-2},\ n \neq 0, 3;$

11) $A_4 \sim A_{4.5},\ A_4 = \langle t\partial_t + x\partial_x, \partial_t - \frac{k}{2-n+m} x^{-1} u\partial_u, |x|^{\frac{1}{2}(1-n+\beta)} u\partial_u,$
$|x|^{\frac{1}{2}(1-n-\beta)} u\partial_u \rangle, \quad A = nx^{-1},\ B = mx^{-2},$
$n \neq 0, 2,\ m \neq n-2,\ m < \frac{1}{4}(n-1)^2, \quad \beta = \sqrt{(n-1)^2 - 4m};$

12) $A_4 \sim A_{4.2},\ A_4 = \langle t\partial_t + x\partial_x, \partial_t + \frac{k}{3-n} x^{-1} \ln |x| u\partial_u, x^{-1} u\partial_u, |x|^{2-n} u\partial_u \rangle,$

$A = nx^{-1},\ B = (n-2)x^{-2},\ n \neq 0, 2, 3;$

13) $A_4 \sim A_{4.6},\ A_4 = \langle t\partial_t + x\partial_x, \partial_t - \frac{k}{2-n+m} x^{-1} u\partial_u,$
$|x|^{\frac{1}{2}(1-n)} u \cos(\beta \ln |x|)\partial_u, |x|^{\frac{1}{2}(1-n)} u \sin(\beta \ln |x|)\partial_u \rangle,$
$A = nx^{-1},\ B = mx^{-2}, n \neq 0,\ m \neq 0,\ m > \frac{1}{4}(n-1)^2,$
$\beta = \sqrt{m - \frac{1}{4}(n-1)^2};$

III. $D = kt,\ k > 0,$

1) $A_4 \sim A_{4.1},\ A_4 = \langle \partial_x, \partial_t - \frac{1}{2} k x^2 u\partial_u, xu\partial_u, u\partial_u \rangle,\ A = B = 0;$

2) $A_4 \sim A_{4.3},\ A_4 = \langle \partial_x, \partial_t - kxu\partial_u, e^{-x} u\partial_u, u\partial_u \rangle,\ A = 1, B = 0;$

3) $A_4 \sim A_{3.8} \oplus A_1,\ A_4 = \langle \partial_x, \partial_t - k\beta^{-2} u\partial_u, u\cos(\beta x)\partial_u, u \sin(\beta x)\partial_u \rangle,$

$A = 0,\ B = \beta^2,\ \beta \neq 0;$

4) $A_4 \sim A_{3.6} \oplus A_1,\ A_4 = \langle \partial_x, \partial_t + k\beta^{-2} u\partial_u, u\mathrm{ch}(\beta x)\partial_u, u \sinh(\beta x)\partial_u \rangle,$

$A = 0,\ B = -\beta^2,\ \beta \neq 0;$

5) $A_4 \sim A_{3.4} \oplus A_1,\ A_4 = \langle \partial_x, \partial_t - 4ku\partial_u, \exp\left(-\frac{1}{2}x\right) u\partial_u, x \exp\left(-\frac{1}{2}x\right) u\partial_u \rangle,$
$A = 1,\ B = \frac{1}{4};$

6) $A_4 \sim A_{3.7} \oplus A_1,\ A_4 = \langle \partial_x, \partial_t - km^{-1} u\partial_u, \exp\left(-\frac{1}{2}(1-\beta)x\right) u\partial_u, \exp\left(-\frac{1}{2}(1+\beta)x\right) u\partial_u \rangle,$

$A = 1,\ B = m,\ m < \frac{1}{4},\ m \neq 0,\ \beta = \sqrt{1-4m};$

7) $A_4 \sim A_{3.9} \oplus A_1,\ A_4 = \langle \partial_x, \partial_t - km^{-1} u\partial_u, \exp\left(-\frac{1}{2}x\right) \cos(\beta x) u\partial_u,$
$\exp\left(-\frac{1}{2}x\right) \sin(\beta x) u\partial_u \rangle,$

$A = 1,\ B = m,\ m > \frac{1}{4},\ \beta = \sqrt{m - \frac{1}{4}};$

IV. $D = kt^{-2},\ k \neq 0,$

$A_4 \sim A_{4.8}\ (q = -1),\ A_4 = \langle \partial_x, t\partial_t + x\partial_x, xu\partial_u, u\partial_u \rangle,\ A = B = 0;$

V. $D = m(x-kt)^{-2},\ k > 0,\ m \neq 0,$

$A_4 \sim A_{4.8}\ (q = -1),\ A_4 = \langle \partial_t + k\partial_x, t\partial_t + x\partial_x, xu\partial_u, u\partial_u \rangle,\ A = B = 0.$

Proof. According to Lemma 5 to get the list of inequivalent equations of the form (6.1) we need to analyze the cases when either $D = 0$ or D does not vanish identically and is obtained through equations (6.8).

If $D = 0$, then function h satisfies equation (6.3) and functions A, B are given by one of the formulas (6.5). It follows from (6.2) that the highest possible dimension of an invariance algebra admitted by (6.1) equals to five. An equation admitting this algebra is equivalent to the following one:

$$u_{tt} = u_{xx} - u^{-1} u_x^2. \tag{6.10}$$

This proves the first part of the assertion of theorem.

The remaining expressions for the functions A, B from (6.5) yield the fourteen items of the case I of the second assertion of theorem.

The fourth expression for the function D given in (6.8) within the equivalence relation (6.7) boils down to either $D = ktx^{-3},\ k > 0$ or $D = kt,\ k > 0$. Here the function h satisfy one of the equations

$$h'' + Ah' + Bh = -\lambda_1 H,$$

where $H = kx^{-3}$ or $H = k$. Analyzing of the corresponding expressions for functions A, B (6.5) results in the cases listed in the cases II and III.

Next, the function D given by the third formula from (6.8) simplifies to $D = kt^{-2}$ $(k \neq 0)$, whence we get the results listed in the case IV of the second assertion of theorem. Similarly, the second expression for the function D gives rise to formulas of the case V.

The first expression for D from (6.8) gives no new invariant equations.

What is left is to prove that the so obtained invariant equations are inequivalent. We omit the proof of this fact.

Theorem is proved.

6.2 Nonlinear equations (2.7) invariant under three-dimensional Lie algebras

The class of PDEs (2.7) does not contain an equation whose invariance algebra is isomorphic to a Lie algebra with a non-trivial Levi ideal (Theorem 12). That is why, to complete the second step of our classification algorithm it suffices to consider three-dimensional solvable real Lie algebras. We begin by considering two decomposable three-dimensional solvable Lie algebras.

Note that while classifying invariant equations (2.7) we skip those belonging to the class (6.1), since the latter has already been analyzed.

6.2.1 Invariance under decomposable Lie algebras

As $A_{3.1} = 3A_1 = A_{2.1} \oplus A_1,\ A_{3.2} = A_{2.2} \oplus A_1$, to construct all realizations of $A_{3.1}$ it suffices to compute all possible extensions of the (already known) realizations of the algebras $A_{2.1} = \langle e_1, e_2 \rangle$ and $A_{2.2} = \langle e_1, e_2 \rangle$. To this end we need to supplement the latter by a basis operator e_3 of the form (2.4) in order to satisfy the commutation relations

$$[e_1, e_3] = [e_2, e_3] = 0. \tag{6.11}$$

What is more, to simplify the form of e_3 we may use those transformations from $\mathcal{E}$ that do not alter the remaining basis operators of the corresponding two-dimensional Lie algebras.

We skip the full calculation details and give several examples illustrating the main calculation steps needed to extend $A_{2.1}$ to a realization of $A_{3.1}$.

Consider the realization $A^1_{2.1}$. Upon checking commutation relations (6.11), where e_3 is of form (2.4), we get

$$\lambda_1 = \lambda_2 = r(t, x) = 0, \quad h = k = \text{const}.$$

Consequently, e_3 is the linear combination of e_1, e_2, namely, $e_3 = \lambda e_1 + k e_2$, which is impossible by the assumption that the algebra under study is three-dimensional. Hence we conclude that the above realization of $A^1_{2.1}$ cannot be extended to a realization of the algebra $A_{3.1}$.

Turn now to the realization $A^2_{2.1}$. Checking commutation relations (6.11), where e_3 is of form (2.4) yields the following realization of $A_{3.1}$:

$$\langle t\partial_t + x\partial_x, \sigma(\xi)\partial_u, \gamma(\xi)\partial_u \rangle, \quad \xi = tx^{-1},$$

where $\gamma'\sigma - \gamma\sigma' \neq 0$. However, the corresponding invariant equation (2.7) is linear.

Finally, consider the realization $A^3_{2.1}$. Inserting its basis operators and the operator e_3 of the form (2.4) into (6.11) and solving the obtained equations gives the following realization of $A_{3.1}$:

$$\langle \partial_t, \partial_x, u\partial_u \rangle.$$

Inserting the obtained coefficients for e_3 into the classifying equation (2.5) we get invariant equation

$$u_{tt} = u_{xx} + uG(\omega), \quad \omega = u^{-1}u_x,$$

where (to ensure non-linearity) we need to have $G_{\omega\omega} \neq 0$.

Similar analysis of the realizations $A^i_{2.1}$ ($i = 4, 5, \ldots, 12, 14$) yields three new invariant equations. For two of thus obtained $A_{3.1}$-invariant equations the corresponding three-dimensional algebras are maximal. The other two may admit four-dimensional invariance algebras provided arbitrary elements are properly specified.

Handling in a similar way the extensions of $A_{2.2}$ up to realizations of $A_{3.2}$ gives ten inequivalent nonlinear equations whose maximal invariance algebras are realizations of the three-dimensional algebra $A_{3.2}$ and four inequivalent equations (2.7) admitting four-dimensional symmetry algebras.

We perform analysis of equations admitting four-dimensional algebras in the next sub-section. Here we present the complete list of nonlinear equations (2.7) whose maximal symmetry algebras are realizations of three-dimensional Lie algebras $A_{3.1}$ and $A_{3.2}$.

$A_{3.1}$-invariant equations

$$\begin{aligned}
A^1_{3.1} &= \langle \partial_t, \partial_x, u\partial_u \rangle : \\
&\quad F = uG(\omega), \quad \omega = u^{-1}u_x; \\
A^2_{3.1} &= \langle \partial_x, \varphi(t)\partial_u, \psi(t)\partial_u \rangle : \\
&\quad \sigma = \psi'\varphi - \psi\varphi' \neq 0, \quad \sigma' = 0 : \\
&\quad F = \varphi^{-1}\varphi'' u + G(t, u_x).
\end{aligned}$$

$A_{3.2}$-invariant equations

$$\begin{aligned}
A^1_{3.2} &= \langle \partial_t, \partial_x, e^x u\partial_u \rangle : \\
&\quad F = -u^{-1}u_x^2 - u\ln|u| + uG(\omega), \\
&\quad \omega = u^{-1}u_x - \ln|u| : \\
A^2_{3.2} &= \langle -t\partial_t - x\partial_x, \partial_t + k\partial_x, u\partial_u \rangle \ (k \geq 0) : \\
&\quad F = u\eta^{-2}G(\omega), \quad \eta = x - kt, \\
&\quad \omega = \eta u^{-1}u_x; \\
A^3_{3.2} &= \langle -t\partial_t - x\partial_x + mu\partial_u, \partial_t + k\partial_x, |\eta|^{-m}\partial_u \rangle \\
&\quad (\eta = x - kt, \ k = m = 0 \text{ or } k > 0, m \in R) : \\
&\quad F = m(k^2 - 1)(m + 1)\eta^{-2}u + |\eta|^{-2-m}G(\omega), \\
&\quad \omega = |\eta|^m(mu + \eta u_x); \\
A^4_{3.2} &= \langle \partial_x, e^x u\partial_u, \partial_t + mu\partial_u \rangle \ (m > 0) : \\
&\quad F = -u^{-1}u_x^2 - u_x + uG(\omega), \\
&\quad \omega = u^{-1}u_x - \ln|u| + mt;
\end{aligned}$$

$$
\begin{aligned}
A_{3.2}^5 &= \langle -t\partial_t - x\partial_x, \partial_x, u\partial_u\rangle : \\
&\quad F = ut^{-2}G(\omega), \quad \omega = tu^{-1}u_x; \\
A_{3.2}^6 &= \langle -t\partial_t - x\partial_x, \partial_t + kx^{-1}u\partial_u, u\partial_u\rangle \ (k>0) : \\
&\quad F = 2ktx^{-2}u_x - 2ktx^{-3}u + k^2t^2x^{-4}u + x^{-2}uG(\omega), \\
&\quad \omega = xu^{-1}u_x + ktx^{-1}; \\
A_{3.2}^7 &= \langle -t\partial_t - x\partial_x, \partial_t + kx^{-1}u\partial_u, \exp(ktx^{-1})\partial_u\rangle \ (k>0) : \\
&\quad F = 2ktx^{-2}u_x + (k^2t^2x^{-4} - 2ktx^{-3} + k^2x^{-2})u + \\
&\quad +x^{-2}\exp(ktx^{-1})G(\omega), \quad \omega = \exp(-ktx^{-1})(xu_x + ktx^{-1}u); \\
A_{3.2}^8 &= \langle \frac{1}{2k}(\partial_t + k\partial_x), e^{x+kt}\partial_u, e^{\eta}\partial_u\rangle \ (k>0, \eta = x-kt) : \\
&\quad F = (k^2-1)u + G(\eta,\omega), \quad \omega = u_x - u; \\
A_{3.2}^9 &= \langle \partial_t + f(x)u\partial_u, e^{(1+f(x))t}\partial_u, f(x)e^{f(x)t}\partial_u\rangle : \\
&\quad F = -(tf'' - t^2(f')^2 - (1+f)^2)u - 2tf'u_x + e^{tf}G(x,\omega), \\
&\quad \omega = e^{-tf}(u_x - f'(t+f^{-1})u), \quad f'' + 2f^2 + f = 0, \quad f \neq 0; \\
A_{3.2}^{10} &= \langle k(t\partial_t + x\partial_x), |t|^{k^{-1}}|\xi|^{\frac{k-1}{2k}}\partial_u, |\xi|^{\frac{k-1}{2k}}\partial_u\rangle \ \ (k \neq 0;1) : \\
&\quad F = \Big[\frac{1-k}{k}\xi^2 + \frac{1-k^2}{4k^2}(1-\xi^2)\Big]t^{-2}u + t^{-2}G(\xi,\omega), \\
&\quad \omega = |\xi|^{\frac{k-1}{2k}}\Big[xu_x + \frac{k-1}{2k}u\Big], \quad \xi = tx^{-1}.
\end{aligned}
$$

6.2.2 Invariance under non-decomposable three-dimensional solvable Lie algebras

There exist seven non-decomposable three-dimensional solvable Lie algebras over the field of real numbers. All those algebras contain a subalgebra which is the two-dimensional Abelian ideal. Consequently, we can use the results of classification of $A_{2.1}$-invariant equations in order to describe equations admitting non-decomposable three-dimensional solvable real Lie algebras. We remind that equations of the form (6.1) has already been analyzed and therefore are not considered in the sequel.

As an example, we perform extension of the realization $A_{2.1}^{10}$ to all possible realizations of non-decomposable three-dimensional solvable real Lie algebras. The remaining realizations are handled in a similar way.

It is straightforward to verify that transformations

$$\bar{t} = \gamma t + \gamma_1, \qquad \bar{x} = \epsilon\gamma x + \gamma_2, \qquad v = \rho(x)u + \theta(x), \tag{6.12}$$

where $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma \neq 0$, $\epsilon = \pm 1$, $\rho \neq 0$ are equivalence transformations for the realization $A_{2.1}^{10} = \langle \partial_t, f(x)u\partial_u\rangle$ $(f \neq 0)$. That is why, we may use the above transformation in order to simplify the form of operator e_3. As a result we get three inequivalent expressions for e_3

1) $e_3 = t\partial_t + x\partial_x + r(t,x)\partial_u$ ($r_t \neq 0$ or $r = 0$);

2) $e_3 = \partial_x + r(t,x)\partial_u$ ($r_t \neq 0$ or $r = 0$);

3) $e_3 = r(t,x)\partial_u$ ($r_t \neq 0$ or $r = 1$).

Let $e_1 = \partial_t,\ e_2 = f(x)u\partial_u$ and $e_3 = t\partial_t + x\partial_x + r(t,x)\partial_u$, then

$$[e_1, e_3] = \partial_t + r_t\partial_u, \quad [e_2, e_3] = -xf'u\partial_u - rf\partial_u.$$

Analyzing commutation relations for the algebras $A_{3.i}$ $(i = 3, 4, \ldots, 9)$ we obtain that the necessary conditions for $A^{10}_{2.1}$ to admit extension to a realization of $A_{3.5}$ read as $r = 0,\ xf' = -f$, of $A_{3.6}$ as $r = 0,\ xf' = f$, and of $A_{3.7}$ as $r = 0,\ xf' = -qf$ $(0 < |q| < 1)$. So $A^{10}_{2.1}$ gives rise to the following realizations:

$$\begin{aligned}
A_{3.5} &: \ e_1 = \partial_t,\ e_2 = x^{-1}u\partial_u,\ e_3 = t\partial_t + x\partial_x;\\
A_{3.6} &: \ e_1 = \partial_t,\ e_2 = xu\partial_u,\ e_3 = t\partial_t + x\partial_x;\\
A_{3.7} &: \ e_1 = \partial_t,\ e_2 = |x|^{-q}u\partial_u,\ e_3 = t\partial_t + x\partial_x \ (0 < |q| < 1).
\end{aligned}$$

If $e_3 = \partial_x + r(t,x)\partial_u$, then

$$[e_1, e_3] = r_t\partial_u, \quad [e_2, e_3] = -f'u\partial_u - rf\partial_u.$$

Analyzing commutation relations for $A_{3.i}$ $(i = 3, 4, \ldots, 9)$ we come to conclusion that the realization $A^{10}_{2.1}$ cannot be extended to a realization of the above three-dimensional Lie algebras.

The same conclusion holds true when $e_3 = r(t,x)\partial_u$ $(r_t \neq 0$ or $r = 1)$.

Let $e_1 = f(x)u\partial_u,\ e_2 = \partial_t$. If $e_3 = t\partial_t + x\partial_x + r(t,x)\partial_u$ $(r_t \neq 0$ or $r = 0)$, then it follows from commutation relations

$$[e_1, e_3] = -(rf + xf'u)\partial_u, \quad [e_2, e_3] = \partial_t + r_t\partial_u$$

that the only possible extension of the realization $A^{10}_{2.1}$ is the realization of $A_{3.5}$:

$$\langle x^{-1}u\partial_u, \partial_t, t\partial_t + x\partial_x\rangle.$$

This realization coincide within notation with the already obtained one.

Next, if $e_3 = \partial_x + r(t,x)\partial_u$ $(r_t \neq 0$ or $r = 0)$, then

$$[e_1, e_3] = -(f'u + rf)\partial_u, \quad [e_2, e_3] = r_t\partial_u.$$

Analyzing commutation relations for $A_{3.i}$ $(i = 3, 4, \ldots, 9)$ we come to conclusion that the realization $A^{10}_{2.1}$ cannot be extended to a realization of the above three-dimensional Lie algebras.

The same conclusion holds true for the case $e_3 = r(t,x)\partial_u$ $(r_t \neq 0$ or $r = 1)$.

Summing up the above considerations we see that the realization $A^{10}_{2.1}$ can be extended to the following realizations of non-decomposable three-dimensional solvable real Lie algebras:

$$\begin{aligned}
&L^1 \sim A_{3.5}, \quad L^1 = \langle \partial_t, x^{-1}u\partial_u, t\partial_t + x\partial_x\rangle;\\
&L^2 \sim A_{3.6}, \quad L^2 = \langle \partial_t, xu\partial_u, t\partial_t + x\partial_x\rangle;\\
&L^3 \sim A_{3.7}, \quad L^3 = \langle \partial_t, |x|^{-q}u\partial_u, t\partial_t + x\partial_x\rangle \ (0 < |q| < 1).
\end{aligned}$$

Solving the corresponding classifying equations yields the following invariant equations:

$$\begin{aligned}
L^1 &: \ u_{tt} = u_{xx} - u^{-1}u_x^2 - 2x^{-2}u\ln|u| + x^{-2}uG(\omega), \quad \omega = xu^{-1}u_x + \ln|u|;\\
L^2 &: \ u_{tt} = u_{xx} - u^{-1}u_x^2 + x^{-2}uG(\omega),\\
&\quad\ \omega = xu^{-1}u_x - \ln|u|;\\
L^3 &: \ u_{tt} = u_{xx} - u^{-1}u_x^2 - q(q+1)x^{-2}u\ln|u| + ux^{-2}G(\omega),\\
&\quad\ \omega = xu^{-1}u_x + q\ln|u| \quad (0 < |q| < 1).
\end{aligned}$$

Note that the algebras L^1, L^2, L^3 are maximal (in Lie's sense) invariance algebras of the corresponding equations.

While classifying nonlinear equations invariant non-decomposable three-dimensional solvable Lie algebras we discovered equations whose maximal invariance algebras are four-dimensional. For example, after extending the realization $A^9_{2.1}$ up to a realization of the algebra $A_{3.3}$ we got the following realization of the latter:

$$\langle \partial_u, \partial_t, \partial_x + t\partial_u \rangle.$$

The corresponding invariant equation (2.7) reads as $u_{tt} = u_{xx} + G(u_x)$. However, the maximal invariance algebra of the above equation is the four-dimensional Lie algebra $\langle \partial_t, t\partial_u, \partial_u, \partial_x \rangle$, which is a realization of $A_{3.3} \oplus A_1$. Note also that we have obtained the above invariant equation when classifying $A_{3.1}$-invariant equations.

By the above reason, we give below only those nonlinear invariant equations whose maximal symmetry algebras are three-dimensional non-decomposable solvable real Lie algebras.

$A_{3.3}$-invariant equations

$$\begin{aligned}
A^1_{3.3} &= \langle u\partial_u, \partial_t + k\partial_x, m\partial_t + k^{-1}xu\partial_u \rangle \quad (k > 0, m \neq 0): \\
& \quad F = -u^{-1}u_x^2 + uG(\omega), \quad \omega = x - kt + mk^2u^{-1}u_x; \\
A^2_{3.3} &= \langle u\partial_u, \partial_x, m\partial_t + xu\partial_u \rangle \ (m > 0): \\
& \quad F = -u^{-1}u_x^2 + uG(\omega), \quad \omega = t - mu^{-1}u_x; \\
A^3_{3.3} &= \langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u, t\partial_t + x\partial_x + \frac{1}{2}u\partial_u \rangle: \\
& \quad F = -\frac{1}{4}t^{-2}u + u_x^3G(\xi, \omega), \ \xi = tx^{-1}, \omega = xu_x^2; \\
A^4_{3.3} &= \langle \partial_u, -t\partial_u, \partial_t + k\partial_x \rangle \ (k \geq 0); \\
& \quad F = G(\eta, u_x), \ \eta = x - kt.
\end{aligned}$$

$A_{3.4}$-invariant equations

$$\begin{aligned}
A^1_{3.4} &= \langle |\eta|^{m-1}\partial_u, \partial_t + k\partial_x, t\partial_t + x\partial_x + (mu + t|\eta|^{m-1})\partial_u \rangle \\
& \quad (\eta = x - kt, \quad k > 0, m \neq 1): \\
& \quad F = (k^2 - 1)(m-1)(m-2)\eta^{-2}u - 2k(m-1)\eta^{m-2}\ln|\eta| \\
& \quad +|\eta|^{m-2}G(\omega), \quad \omega = [\eta u_x - (m-1)u]|\eta|^{-m}; \\
A^2_{3.4} &= \langle \partial_u, -t\partial_u, \partial_t + k\partial_x + u\partial_u \rangle \ (k \geq 0): \\
& \quad F = e^tG(\eta, \omega), \quad \eta = x - kt, \ \omega = e^{-t}u_x; \\
A^3_{3.4} &= \langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u, t\partial_t + x\partial_x + \frac{3}{2}u\partial_u \rangle: \\
& \quad F = -\frac{1}{4}t^{-2}u + u_x^{-1}G(\xi, \omega), \ \xi = tx^{-1}, \quad \omega = x^{-1}u_x^2; \\
A^4_{3.4} &= \langle kx^{-1}u\partial_u, \partial_t - kx^{-1}\ln|x|u\partial_u, t\partial_t + x\partial_x \rangle \ (k > 0): \\
& \quad F = -3ktx^{-3}u - 2x^{-2}u\ln|u| - u^{-1}u_x^2 + x^{-2}uG(\omega), \\
& \quad \omega = xu^{-1}u_x + \ln|u| + ktx^{-1};
\end{aligned}$$

$$A^5_{3.4} = \langle \exp(ktx^{-1})\partial_u, \partial_t + kx^{-1}u\partial_u, t\partial_t + x\partial_x + (u + t\exp(ktx^{-1}))\partial_u\rangle \ (k>0):$$
$$F = k^2x^{-4}u(t^2+x^2) + 2x^{-1}(ktx^{-1}+1)u_x$$
$$+2k\exp(ktx^{-1})x^{-1}\ln|x| + x^{-1}\exp(ktx^{-1})G(\omega),$$
$$\omega = \exp(-ktx^{-1})(u_x + ktx^{-2}u).$$

$A_{3.5}$-invariant equations

$$A^1_{3.5} = \langle |\eta|^{m-1}\partial_u, \partial_t + k\partial_x, t\partial_t + x\partial_x + mu\partial_u\rangle \ (k>0,\ m\neq 1)$$
$$F = (k^2-1)(m-1)(m-2)\eta^{-2}u + |\eta|^{m-2}G(\omega),$$
$$\omega = |\eta|^{-m}[\eta u_x - (m-1)u], \eta = x - kt;$$
$$A^2_{3.5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x\rangle :$$
$$F = u_x^2 G(u);$$
$$A^3_{3.5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x + mu\partial_u\rangle \ (m\neq 0):$$
$$F = |u|^{1-\frac{2}{m}}G(\omega),\ \omega = |u_x|^m|u|^{1-m};$$
$$A^4_{3.5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x + \partial_u\rangle :$$
$$F = e^{-2u}G(\omega),\ \omega = e^u u_x;$$
$$A^5_{3.5} = \langle \partial_t, x^{-1}u\partial_u, t\partial_t + x\partial_x\rangle :$$
$$F = -u^{-1}u_x^2 - 2x^{-2}u\ln|u| + x^{-2}uG(\omega),$$
$$\omega = xu^{-1}u_x + \ln|u|;$$
$$A^6_{3.5} = \langle \partial_t + kx^{-1}u\partial_u, \exp(ktx^{-1})\partial_u, t\partial_t + x\partial_x + u\partial_u\rangle \ (k>0):$$
$$F = kx^{-4}u[kt^2 - 2tx + kx^2] + 2ktx^{-2}u_x + x^{-1}\exp(ktx^{-1})G(\omega),$$
$$\omega = \exp(-ktx^{-1})(u_x + ktx^{-2}u);$$
$$A^7_{3.5} = \langle \varphi(t)\partial_u, \psi(t)\partial_u, \partial_x + u\partial_u\rangle \ (\varphi'\psi - \varphi\psi' \neq 0):$$
$$F = \varphi^{-1}\varphi'' u + u_x G(t,\omega),$$
$$\omega = e^{-x}u_x, \quad \varphi''\psi - \varphi\psi'' = 0.$$

$A_{3.6}$-invariant equations

$$A^1_{3.6} = \langle \partial_t + k\partial_x, |\eta|^{m+1}\partial_u, t\partial_t + x\partial_x + mu\partial_u\rangle \ (k>0,\ m\neq -1):$$
$$F = m(k^2-1)(m+1)\eta^{-2}u + |\eta|^{m-2}G(\omega),$$
$$\omega = |\eta|^{1-m}[u_x - \eta^{-1}(m+1)u], \eta = x - kt;$$
$$A^2_{3.6} = \langle \partial_t + mx^{-1}u\partial_u, xu\partial_u, t\partial_t + x\partial_x\rangle \ (m\geq 0):$$
$$F = -u^{-1}u_x^2 - 2mtx^{-3}u + x^{-2}uG(\omega),$$
$$\omega = xu^{-1}u_x - \ln|u| + 2mtx^{-1};$$

$$A^3_{3.6} = \langle \partial_t + kx^{-1}u\partial_u, \exp(ktx^{-1})\partial_u, t\partial_t + x\partial_x - u\partial_u\rangle \ (k > 0):$$

$$F = x^{-4}[k^2x^2 - 2ktx + k^2t^2]u + 2ktx^{-2}u_x + x^{-3}\exp(ktx^{-1})G(\omega),$$

$$\omega = \exp(-ktx^{-1})(x^2u_x + ktu);$$

$$A^4_{3.6} = \langle e^{-t}\partial_u, e^t\partial_u, \partial_t + k\partial_x\rangle \ (k \geq 0):$$

$$F = u + G(\eta, u_x), \ \ \eta = x - kt;$$

$$A^5_{3.6} = \langle |t|^{-\frac{1}{2}}\partial_u, |t|^{\frac{3}{2}}\partial_u, t\partial_t + x\partial_x + \frac{1}{2}u\partial_u\rangle:$$

$$F = \frac{3}{4}t^{-2}u + |t|^{-\frac{3}{2}}G(\xi, \omega), \ \ \xi = tx^{-1}, \quad \omega = x^{-1}u_x^2.$$

$A_{3.7}$-invariant equations

$$A^1_{3.7} = \langle \partial_t + k\partial_x, |\eta|^{m-q}\partial_u, t\partial_t + x\partial_x + mu\partial_u\rangle$$
$$(k > 0, m \neq q, \ 0 < |q| < 1):$$

$$F = (k^2 - 1)(m - q)(m - q - 1)\eta^{-2}u + |\eta|^{m-2}G(\omega),$$

$$\omega = |\eta|^{1-m}[u_x - (m - q)\eta^{-1}u], \ \ \eta = x - kt;$$

$$A^2_{3.7} = \langle \partial_t + kx^{-1}u\partial_u, \exp(ktx^{-1})\partial_u, t\partial_t + x\partial_x + qu\partial_u\rangle$$
$$(k > 0, \ 0 < |q| < 1):$$

$$F = [k^2x^{-2} + k^2x^{-4}t^2 - 2ktx^{-3}]u + 2ktx^{-2}u_x$$
$$+|x|^{q-2}\exp(ktx^{-1})G(\omega),$$

$$\omega = |x|^{1-q}\exp(-ktx^{-1})(u_x + ktx^{-2}u);$$

$$A^3_{3.7} = \langle |t|^{\frac{1}{2}q}\partial_u, |t|^{1-\frac{1}{2}q}\partial_u, t\partial_t + x\partial_x + (1 + \frac{1}{2}q)u\partial_u\rangle \ (q \neq 0, \pm 1):$$

$$F = \frac{1}{4}q(q - 2)t^{-2}u + |t|^{\frac{1}{2}(q-2)}G(\xi, \omega),$$

$$\xi = tx^{-1}, \ \ \omega = |t|^{-\frac{1}{2}q}u_x;$$

$$A^4_{3.7} = \langle \exp\left(\frac{1}{2}(q-1)t\right)\partial_u, \exp\left(\frac{1}{2}(1-q)t\right)\partial_u, \partial_t + k\partial_x + \frac{1}{2}(1+q)u\partial_u\rangle$$
$$(q \neq 0, \pm 1; k \geq 0):$$

$$F = \frac{1}{4}(q-1)^2u + \exp\left(\frac{1}{2}(1+q)t\right)G(\eta, \omega),$$

$$\eta = x - kt, \ \ \omega = \exp\left(-\frac{1}{2}(1+q)t\right)u_x;$$

$$A^5_{3.7} = \langle \partial_t + kx^{-1}u\partial_u, |x|^{-q}u\partial_u, t\partial_t + x\partial_x\rangle \ (k \geq 0, q \neq 0, \pm 1):$$

$$F = -u^{-1}u_x^2 - q(q+1)x^{-2}u\ln|u| + k(q-1)(q+2)tx^{-3}u$$

$$+ux^{-2}G(\omega), \quad \omega = xu^{-1}u_x + q\ln|u| + k(1-q)tx^{-1}.$$

$A_{3.8}$-invariant equations

$$A^1_{3.8} = \langle \cos t\partial_u, -\sin t\partial_u, \partial_t + k\partial_x\rangle \ (k \geq 0):$$

$$F = -u + G(\eta, u_x), \eta = x - kt;$$

$$
\begin{aligned}
A^2_{3.8} &= \langle |t|^{\frac{1}{2}}\cos(\ln|t|)\partial_u, -|t|^{\frac{1}{2}}\sin(\ln|t|)\partial_u, t\partial_t + x\partial_x + \frac{1}{2}u\partial_u\rangle : \\
& F = -\frac{5}{4}t^{-2}u + |t|^{-\frac{3}{2}}G(\xi,\omega), \\
& \xi = tx^{-1},\ \omega = |t|^{\frac{1}{2}}u_x.
\end{aligned}
$$

$A_{3.9}$-invariant equations

$$
\begin{aligned}
A^1_{3.9} &= \langle \sin t\partial_u, \cos t\partial_u, \partial_t + k\partial_x + qu\partial_u\rangle\ (k \geq 0, q > 0) : \\
& F = -u + e^{qt}G(\eta,\omega),\ \eta = x - kt, \quad \omega = e^{-qt}u_x; \\
A^2_{3.9} &= \langle |t|^{\frac{1}{2}}\sin(\ln|t|)\partial_u, |t|^{\frac{1}{2}}\cos(\ln|t|)\partial_u, t\partial_t + x\partial_x + (\frac{1}{2} + q)u\partial_u\rangle \\
& (q \neq 0) : F = -\frac{5}{4}t^{-2}u + |t|^{q-\frac{3}{2}}G(\xi,\omega), \\
& \xi = tx^{-1}, \quad \omega = |t|^{\frac{1}{2}-q}u_x.
\end{aligned}
$$

6.3 Complete group classification of equation (2.7)

The aim of this subsection is finalizing group classification of (2.7). The majority of invariant equations obtained in the previous subsection contain arbitrary functions of one variable. So that we can utilize the standard Lie-Ovsyannikov approach in order to complete their group classification.

6.3.1 Equations depending on an arbitrary function of one variable.

Note that equations belonging to the already investigated class of (6.1) are not considered.

As our computations show, new results could be obtained for the equations

$$
\begin{aligned}
u_{tt} &= u_{xx} + uG(\omega), \quad \omega = u^{-1}u_x, && (6.13) \\
u_{tt} &= u_{xx} + G(u_x) && (6.14)
\end{aligned}
$$

only. Below we give (without proof) the assertions describing their group properties.

Assertion 7 *Equation (6.13) admits wider symmetry group iff it is equivalent to the following equation*

$$u_{tt} = u_{xx} + mu^{-1}u_x^2 \qquad (m \neq 0, -1). \tag{6.15}$$

The maximal invariance algebra of (6.15) is the four-dimensional Lie algebra

$$A_4 \sim A_{3.5} \oplus A_1, \quad A_4 = \langle \partial_t, \partial_x, t\partial_t + x\partial_x, u\partial_u\rangle.$$

Assertion 8 *Equation (6.14) admits wider symmetry group iff it is equivalent to one of the following PDEs:*

$$
\begin{aligned}
u_{tt} &= u_{xx} + e^{u_x}; && (6.16) \\
u_{tt} &= u_{xx} + m\ln|u_x|, \quad m > 0; && (6.17) \\
u_{tt} &= u_{xx} + |u_x|^k, \quad k \neq 0, 1. && (6.18)
\end{aligned}
$$

The maximal invariance algebras of the above equations are five-dimensional solvable Lie algebras listed below.

$$
\begin{aligned}
A^2_5 &= \langle \partial_t, \partial_x, \partial_u, t\partial_u, t\partial_t + x\partial_x + (u - x)\partial_u\rangle; \\
A^3_5 &= \langle \partial_t, \partial_x, \partial_u, t\partial_u, t\partial_t + x\partial_x + (2u + \frac{1}{2}mt^2)\partial_u\rangle; \\
A^4_5 &= \langle \partial_t, \partial_x, \partial_u, t\partial_u, t\partial_t + x\partial_x + \frac{k-2}{k-1}u\partial_u\rangle.
\end{aligned}
$$

Analyzing the remaining equations containing arbitrary functions of one variable we come to conclusion that one of them can admit wider invariance groups iff either

1) it is equivalent to PDE of the form (6.1), or

2) it is equivalent to PDE of the form (6.15).

Skipping the proof, we present two typical examples. We begin with the equation

$$u_{tt} = u_{xx} + u + G(u_x). \tag{6.19}$$

This equation is invariant under the four-dimensional algebra $\langle \partial_t, \partial_x, e^t\partial_u, e^{-t}\partial_u\rangle$ isomorphic to $A_{3.6} \oplus A_1$. Inserting $F = u + G(u_x)$ into classifying equation (2.5) yields the system of two equations for the function G

$$h'G' = -h'' - 2\lambda, \quad [(h-\lambda)u_x + r_x]G' - (h - 2\lambda)G = r_{tt} - r_{xx} - 2h'u_x - r.$$

As we require $G'' \neq 0$, it follows from the first equation that $\lambda = h' = 0$ and the second equation takes the form

$$(hu_x + r_x)G' - hG = r_{tt} - r_{xx} - r.$$

Upon differentiating the above equation twice with respect to u_x we get $(hu_x + r_x)G'' = 0$. As $G'' \neq 0$, relations $h = r_x = 0$ hold. Hence we conclude that the class of PDEs (6.19) does not contain equations admitting five-dimensional algebras.

The system of determining equations for symmetry group of $A^2_{3.2}$-invariant equation

$$u_{tt} = u_{xx} + u\eta^{-2}G(\omega), \quad \eta = x - kt, \quad \omega = \eta u^{-1}u_x, \quad k \geq 0, \tag{6.20}$$

read as

$$\begin{aligned}
&(\eta^{-2}r\omega - \eta^{-1}r_x)G_\omega - \eta^{-2}rG = r_{tt} - r_{xx},\\
&[(\lambda_2 - k\lambda_1)\eta^{-3}\omega + \eta^{-1}h']G_\omega - 2(\lambda_2 - k\lambda_1)\eta^{-3}G = -2h'\eta^{-1}\omega - h''.
\end{aligned}$$

Differentiating the first equation with respect to ω yields

$$(\eta^{-2}r\omega - \eta^{-1}r_x)G_{\omega\omega} = 0,$$

whence in view of inequality $G_{\omega\omega} \neq 0$ we get $r = 0$.

Next, differentiating the second equation twice by ω we get

$$[(\lambda_2 - lk\lambda_1)\eta^{-3}\omega + \eta^{-1}h']G_{\omega\omega\omega} = 0,$$

whence it follows that $G_{\omega\omega\omega} = 0$. Indeed, if this relation does not hold, we have $\lambda_2 = k\lambda_1$, $h' = 0$ and operator (2.4) takes the form

$$\lambda(t\partial_t + x\partial_x) + \lambda_1(\partial_t + k\partial_x) + C_1u\partial_u, \quad \lambda, \lambda_1 C_1 \in \mathbb{R}, \quad k \geq 0.$$

As the above operator contains at most three arbitrary constants it cannot generate a four-parameter Lie transformation group.

By the above argument we can restrict our considerations to the following class of functions G :

$$G = A\omega^2 + B\omega + C, \quad A \neq 0, -1, \; B, C \in \mathbb{R}. \tag{6.21}$$

We can suppose that $A \neq -1$ in (6.21) (since otherwise (6.20) belongs to the class of PDEs (6.1)). Inserting (6.21) into the second equation from (6.21) yields

$$2(A+1)\eta^2 h' = B(\lambda_2 - k\lambda_1), \quad \eta^2 Bh' + \eta^3 h'' = 2C(\lambda_2 - k\lambda_1). \tag{6.22}$$

If $k > 0$, then this system splits into the following equations (note that $h = h(x)$):

$$h' = 0, \quad B(\lambda_2 - k\lambda_1) = C(\lambda_2 - k\lambda_1) = 0.$$

Provided $|B| + |C| \neq 0$, there is no way for extending symmetry of equation (6.20). If, on the contrary, $B = C = 0$, then $F = Au^{-1}u_x^2 (A \neq 0, -1)$ and we obtain the equation equivalent to (6.15). Under $k = 0$ system (6.22) takes the form

$$2(A+1)x^2h' = \lambda_2 B, \quad x^2Bh' + h''x^3 = 2\lambda_2 C.$$

Hence

$$h = -\frac{1}{2}\lambda_2(A+1)^{-1}Bx^{-1} + C_1, \quad C_1 \in \mathbb{R}, \quad C = \frac{B^2 - 2B}{4(A+1)}.$$

In this case equation (6.20) does admit additional symmetry operator

$$\partial_x - \frac{B}{2(A+1)}x^{-1}u\partial_u$$

but the change of variables

$$\bar{t} = t, \qquad \bar{x} = x, \qquad u = |x|^{\nu}v, \qquad \nu = -\frac{B}{2(A+1)},$$

reduces it to the form (6.15).

So equation (6.20) admits wider symmetry group iff it is either belongs to the class of (6.1) or is equivalent to (6.15).

To finalize the procedure of group classification of equations (2.7) we need to consider invariant equations obtained in the previous section that contain arbitrary functions of two variables.

6.3.2 Classification of equations with arbitrary functions of two variables.

In the case under study the standard Lie-Ovsyannikov method is inefficient and we apply our classification algorithm. In order to do this we perform extension of three-dimensional solvable Lie algebras to all possible realizations of four-dimensional solvable Lie algebras. The next step will be to check which of the obtained realizations are symmetry algebras of nonlinear equations of the form (2.7). In what follows we use the results of the paper [45], where all inequivalent (within the action of inner automorphism group) four-dimensional solvable abstract Lie algebras are given.

We give full computation details for the case of $A_{3.6}$-invariant equations. As shown in the previous subsection, there are two inequivalent $A_{3.6}$-invariant equations, namely,

$$\begin{aligned}
A^4_{3.6} &= \langle e^{-t}\partial_u, e^t\partial_u, \partial_t + k\partial_x\rangle \\
&\quad (k \geq 0) : F = u + G(\eta, u_x), \ \eta = x - kt; \\
A^5_{3.6} &= \langle |t|^{-\frac{1}{2}}\partial_u, \ |t|^{\frac{3}{2}}\partial_u, t\partial_t + x\partial_x + \frac{1}{2}u\partial_u\rangle : \\
&\quad F = \frac{3}{4}t^{-2}u + |t|^{-\frac{3}{2}}G(\xi, \omega), \ \xi = tx^{-1}, \ \omega = x^{-1}u_x^2.
\end{aligned}$$

According to [45] the algebra $A_{3.6}$ is the subalgebra of the following four-dimensional solvable Lie algebras: $2A_{2.2}, A_{3.6} \oplus A_1; A_{4.2}(q = -1); A_{4.8}(q = -\frac{1}{2})$.

Algebra $2A_{2.2}$. The algebra $2A_{2.2} = \langle e_1, e_2, e_3, e_4\rangle$ is determined by the following commutation relations (note that we give non-zero relations only):

$$[e_1, e_2] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = 0, \quad [e_1, e_2] = e_2, [e_3, e_4] = e_4.$$

It contains a subalgebra $A_{3.6} = \langle e_1 - e_3, e_2, e_4\rangle$. That is why, we can choose as $e_1 - e_3, e_2, e_4$ the basis operators of the realization of $A_{3.6}$. Next, we take as $e_1 + e_3$ an arbitrary operator of the form (2.4) and require for the commutation relations

$$[e_1 - e_3, e_1 + e_3] = 0, \quad [e_1 + e_3, e_2] = e_2, \quad [e_1 + e_3, e_4] = e_4 \tag{6.23}$$

to hold.

Realization $A^4_{3.6}$. In this case

$$e_1 - e_3 = -\partial_t - k\partial_x, \quad e_2 = e^{-t}\partial_u, \quad e_4 = e^t\partial_u,$$
$$e_1 + e_3 = (\lambda t + \lambda_1)\partial_t + (\lambda x + \lambda_2)\partial_x + (hu + r)\partial_u.$$

It follows from (6.23) that

$$\lambda = \lambda_1 = 0, \quad r = \gamma = \gamma(\eta), \quad h = -1.$$

Using the change of variables

$$\bar{t} = t, \quad \bar{x} = x, \quad v = u + \Lambda(\eta),$$

where $\Lambda = \Lambda(\eta)$ is a solutions of equation $\lambda_2\Lambda' + \Lambda = -\gamma$, we simplify the operator $e_1 + e_3$ to become

$$e_1 + e_3 = \alpha\partial_x - u\partial_u, \quad \alpha \in \mathbb{R}.$$

Requiring invariance under the above operator yields that $\alpha \neq 0$ (otherwise G would be linear in u_x). With this condition we rewrite the invariant equation to become

$$G = \exp(-\alpha^{-1}\eta)H(\omega), \qquad \omega = \exp(\alpha^{-1}\eta)u_x.$$

Thus we arrive at the following realization of the algebra $2A_{2.2}$:

$$\langle e^{-t}\partial_u, e^t\partial_u, \partial_t + k\partial_x, \alpha\partial_x - u\partial_u\rangle \ (k \geq 0, \ \alpha \neq 0).$$

This algebra is admitted by the equation

$$u_{tt} = u_{xx} + u + \exp(-\alpha^{-1}\eta)G(\omega), \ \eta = x - kt, \ \omega = \exp(\alpha^{-1}\eta)u_x.$$

If the function G ($G_{\omega\omega} \neq 0$) is arbitrary, then the obtained realization is the maximal symmetry algebra of the equation under study. What is more, no G exists such that the above equation admits a wider invariance algebra.

Realization $A^5_{3.6}$. In this case

$$e_1 - e_3 = -t\partial_t - x\partial_x - \frac{1}{2}u\partial_u, \quad e_2 = |t|^{-\frac{1}{2}}\partial_u,$$
$$e_1 + e_3 = (\lambda_t + \lambda_1)\partial_t + (\lambda_x + \lambda_2)\partial_x + (hu + r)\partial_u, \quad e_4 = |t|^{\frac{3}{2}}\partial_u.$$

It follows from commutation relations (6.23) that $\lambda_1 = \lambda_2 = \lambda = 0$, $h = -1$, $r = |t|^{\frac{1}{2}}\gamma(\xi)$, $\xi = tx^{-1}$.

Making the change of variables

$$\bar{t} = t, \quad \bar{x} = x, \quad v = u - |t|^{\frac{1}{2}}\gamma(\xi)$$

we get $r = 0$ in $e_1 + e_3$. Consequently, without loss of generality we can choose $e_1 + e_3 = -u\partial_u$. Requiring for $A^5_{3.6}$-invariant equation to admit the operator $e_1 + e_3$ yields the equation $2\omega G_\omega = -G$, whence $G = |\omega|^{-\frac{1}{2}}H(\xi)$. Consequently, the function F is linear in the variable u_x. This means that $A^5_{3.6}$ does not admit extension to a realization $2A_{2.2}$ that can be a symmetry algebra of an equation of the form (2.7).

Algebra $A_{3.6} \oplus A_1$. What we need to do here is to supplement the set of operators e_1, e_2, e_3 forming the basis of $A_{3.6}$ by the operator e_4 of the form (2.4) and verify the commutation relations

$$[e_1, e_4] = [e_2, e_4] = [e_3, e_4] = 0 \tag{6.24}$$

Realization $A^4_{3.6}$. It follows from (6.24) that $h = \lambda = \lambda_1 = 0, \quad r = \gamma(\eta), \ \eta = x - kt$ in the operator e_4 so that

$$e_4 = \alpha\partial_x + \gamma(\eta)\partial_u, \quad \alpha \in \mathbb{R}.$$

If $\alpha \neq 0$, then e_4 is equivalent to ∂_x. Hence we get two possible realizations of the algebra $A_{3.6} \oplus A_1$:

$$\langle e^{-t}\partial_u, \ e^t\partial_u, \partial_t, \partial_x\rangle;$$
$$\langle e^{-t}\partial_u, \ e^t\partial_u, \partial_t + k\partial_x, \gamma(\eta)\partial_u\rangle.$$

Analyzing the above realizations we come to conclusion that the second one cannot be invariance algebra of a nonlinear equation of the form (2.7). The first realization is the maximal (if G is an arbitrary function) invariance algebra of the equation (6.19).

Realization $A^5_{3.6}$. It follows from (6.24) that $\lambda_1 = \lambda_2 = h = \lambda = 0$,

$$r = |t|^{\frac{1}{2}}\gamma(\xi), \quad \xi = tx^{-1},$$

so that the operator e_4 necessarily takes the form $e_4 = |t|^{\frac{1}{2}}\gamma(\xi)\partial_u$. As the straightforward verification shows thus obtained realization cannot be invariance algebra of a nonlinear equation of the form (2.7).

Algebra $A_{4.2}$ $(q = -1)$. We need to supplement the set of operators e_1, e_2, e_4 forming the basis of $A_{3.6}$ by the operator e_3 of the form (2.4) so that the following commutation relation hold

$$[e_1, e_3] = [e_2, e_3] = 0, \quad [e_3, e_4] = e_2 + e_3. \tag{6.25}$$

Realization $A^4_{3.6}$. In this case

$$e_1 \ = \ e^{-t}\partial_u, \quad e_2 = e^t\partial_u, \quad e_4 = -\partial_t - k\partial_x$$

and it follows from (6.25) that coefficients of e_3 satisfy equations $h = \lambda = \lambda_1 = \lambda_2 = 0$, the function r being a solution of the equation

$$r_t + kr_x = r + e^t.$$

This realization cannot be invariance algebra of nonlinear equation of the form (2.7).

Realization $A^5_{3.6}$. In this case

$$e_1 \ = \ |t|^{-\frac{1}{2}}\partial_u, \quad e_2 = |t|^{\frac{3}{2}}\partial_u, \quad e_4 = -t\partial_t - x\partial_x - \frac{1}{2}u\partial_u.$$

It follows from (6.25) that the coefficients of the operator e_3 satisfy equations $\lambda = \lambda_1 = \lambda_2 = h = 0$ and the function r is a solution of the equation

$$tr_t + xr_x = \frac{3}{2}r + |t|^{\frac{3}{2}}.$$

Further analysis shows that the so obtained realization cannot be invariance algebra of a nonlinear equation of the form (2.7).

Algebra $A_{4.8}$ $(q = -\frac{1}{2})$. We need to supplement the set of operators e_1, e_3, e_4 forming the basis of $A_{3.6}$ by the e_2 of the form (2.4) in order to satisfy the commutation relations

$$[e_1, e_2] = 0, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = e_2. \tag{6.26}$$

Realization $A_{3.6}^4$. In this case

$$e_1 = e^{-t}\partial_u, \quad e_3 = e^t\partial_u, \quad e_4 = \frac{1}{2}\partial_t + \frac{1}{2}k\partial_x$$

and the second commutation relation yields the false equality $1 = 0$.

Realization $A_{3.6}^5$. In this case

$$e_1 = |t|^{-\frac{1}{2}}\partial_u, \quad e_2 = |t|^{\frac{3}{2}}\partial_u, \quad e_4 = -t\partial_t - x\partial_x - \frac{1}{2}u\partial_u$$

and again the second commutation relation from (6.26) cannot be satisfied.

So that there are no extensions of the realization of $A_{3.6}$ to a realization of the algebra $A_{3.8}$ ($q = -\frac{1}{2}$).

The remaining equations containing arbitrary functions of two variables are handled in a similar way. The results can be summarized as follows

1) if the functions contained in the equations under study are arbitrary, then the corresponding realizations are their maximal invariance algebras, and

2) except for equation (6.14), all the equations in question do not allow for extension of their symmetry.

Below we give the complete list of invariant equations obtained through group analysis of equations with arbitrary functions of two variables.

6.3.3 Equations invariant under four-dimensional solvable Lie algebras.

$A_{2.2} \oplus 2A_1$-invariant equations

1) $\langle \partial_x, \partial_t + u\partial_u, e^t\partial_u, e^{-t}\partial_u \rangle : \quad F = u + e^t G(\omega), \quad \omega = u^{-t}u_x;$

2) $\langle \frac{1}{2k}(\partial_t + k\partial_x), e^{x+kt}\partial_u, e^{\eta}\partial_u, \partial_x + u\partial_u \rangle \ (k > 0, \eta = x - kt) :$
$F = (k^2 - 1)u + e^{\eta}G(\omega), \quad \omega = e^{-\eta}(u_x - u).$

$2A_{2.2}$-invariant equations

1) $\langle \partial_t + \epsilon u\partial_u, \partial_x, e^{x+kt}\partial_u, e^{x-kt}\partial_u \rangle \ (\epsilon = 0, 1; k > 0) :$
$F = (k^2 - 1)u + e^{\epsilon t}G(\omega), \quad \omega = e^{-\epsilon t}(u_x - u);$

2) $\langle \alpha\partial_x - u\partial_u, \partial_t + k\partial_x, e^{-t}\partial_u, e^t\partial_u \rangle \ (k \geq 0, \alpha > 0) :$
$F = u + \exp(-\alpha^{-1}\eta)G(\omega), \ \eta = x - kt, \ \omega = \exp(\alpha^{-1}\eta)u_x.$

$A_{3.3} \oplus A_1$-invariant equations

1) $\langle \partial_t, \partial_x, \partial_u, t\partial_u \rangle : F = G(u_x).$

$A_{3.4} \oplus A_1$-invariant equations

1) $\langle \partial_u, \partial_x, t\partial_t + x\partial_x + (u + x)\partial_u, t\partial_u \rangle :$
$F = t^{-1}G(\omega), \quad \omega = u_x - \ln|t|;$

2) $\langle \partial_t + u\partial_u, \partial_x, t\partial_u, \partial_u \rangle : \quad F = e^t G(\omega), \quad \omega = e^{-t}u_x;$

3) $\langle x^{-1}\partial_u, \partial_x - x^{-1}(u + \ln|x|)\partial_u, t\partial_t + x\partial_x, tx^{-1}\partial_u \rangle :$
$F = 2x^{-1}u_x + x^{-2} + t^{-1}x^{-1}G(\omega), \ \omega = xu_x + u - \ln|tx^{-1}|.$

$A_{3.5} \oplus A_1$-invariant equations

$$\begin{array}{ll} 1) & \langle \partial_x, \partial_u, t\partial_t + x\partial_x + u\partial_u, t\partial_u\rangle : \quad F = t^{-1}G(u_x); \\ 2) & \langle x^{-1}\partial_u, \partial_x - x^{-1}u\partial_u, t\partial_t + x\partial_x, tx^{-1}\partial_u\rangle : \\ & F = -2x^{-2}u + 2t^{-1}(u_x + x^{-1}u)\ln|t(u_x + x^{-1}u)| \\ & +t^{-1}(u_x + x^{-1}u)G(\omega), \quad \omega = xu_x + u. \end{array}$$

$A_{3.6} \oplus A_1$-invariant equations

$$\begin{array}{ll} 1) & \langle \partial_x, t\partial_u, t\partial_t + x\partial_x, \partial_u\rangle : \quad F = t^{-2}G(\omega), \ \omega = t^{-1}u_x; \\ 2) & \langle \partial_t, \partial_x, e^t\partial_u, e^{-t}\partial_u\rangle : \quad F = u + G(u_x). \end{array}$$

$A_{3.7} \oplus A_1$-invariant equations

$$\begin{array}{ll} 1) & \langle \exp\left(-\frac{1}{2}(1-q)t\right)\partial_u, \exp\left(\frac{1}{2}(1-q)t\right)\partial_u, \partial_t + \frac{1}{2}(1+q)u\partial_u, \partial_x\rangle \\ & (q \neq 0, \pm 1) : \ F = \frac{1}{4}(1-q)^2 u + \exp\left(\frac{1}{2}(1+q)t\right)G(\omega), \\ & \omega = \exp\left(-\frac{1}{2}(1+q)t\right)u_x; \\ 2) & \langle \partial_x, |t|^{\frac{1}{2}(1-q)}\partial_u, |t|^{\frac{1}{2}(1+q)}\partial_u, t\partial_t + x\partial_x + \frac{1}{2}(1+q)u\partial_u\rangle \\ & (q \neq 0, \pm 1) : F = \frac{1}{4}(q^2-1)t^{-2}u + |t|^{\frac{1}{2}(q-3)}G(\omega), \\ & \omega = |t|^{\frac{1}{2}(1-q)}u_x; \\ 3) & \langle |t|^{-\frac{1}{q}}|\xi|^{\frac{q+1}{2q}}\partial_u, \partial_x - \frac{1+q}{2q}x^{-1}u\partial_u, -q(t\partial_t + x\partial_x), |\xi|^{\frac{1+q}{2q}}\partial_u\rangle \\ & (q \neq 0, \pm 1) : F = \left[\frac{1-q^2}{4q^2}(t^{-2} + x^{-2})\right]u + \frac{1+q}{q}x^{-1}u_x \\ & +t^{-2}|\xi|^{\frac{1+q}{2q}}G(\omega), \quad \xi = tx^{-1}, \quad \omega = |\xi|^{\frac{q-1}{2q}}\left[xu_x + \frac{q+1}{2q}u\right]. \end{array}$$

$A_{3.8} \oplus A_1$-invariant equations

$$1) \qquad \langle \sin t\partial_u, \cos t\partial_u, \partial_t, \partial_x\rangle : F = -u + G(u_x).$$

$A_{3.9} \oplus A_1$-invariant equations

$$\begin{array}{ll} 1) & \langle \sin t\partial_u, \cos t\partial_u, \partial_t + qu\partial_u, \partial_x\rangle (q > 0) : \\ & F = -u + e^{qt}G(\omega), \quad \omega = e^{-qt}u_x. \end{array}$$

$A_{4.1}$-invariant equations

$$\begin{array}{ll} 1) & \langle \partial_u, -t\partial_u, \partial_x, \partial_t - tx\partial_u\rangle : \quad F = G(\omega), \quad \omega = u_x + \frac{1}{2}t^2; \\ 2) & \langle \partial_u, -t\partial_u, \alpha\partial_x + \frac{1}{2}t^2\partial_u, \partial_t + kx\partial_x\rangle \ (k \geq 0, \alpha > 0) : \\ & F = \alpha^{-1}(x - kt) + G(u_x). \end{array}$$

$A_{4.2}$-invariant equations

$$1)\qquad \langle |t|^{1-\frac{1}{2}q}\partial_u, |t|^{\frac{1}{2}q}\partial_u, \partial_x, t\partial_t + x\partial_x + \left[\left(1+\frac{1}{2}q\right)u + x|t|^{\frac{1}{2}q}\right]\partial_u\rangle$$

$$(q\neq 0,1): F = \frac{1}{4}q(q-2)t^{-2}u + |t|^{\frac{1}{2}(q-3)}G(\omega),$$

$$\omega = |t|^{\frac{1}{2}(1-q)}u_x - 2|t|^{\frac{1}{2}};$$

$$2)\qquad \langle \partial_x, \sqrt{|t|}\partial_u, \sqrt{|t|}\ln|t|\partial_u, t\partial_t + x\partial_x + \left(q+\frac{1}{2}\right)u\partial_u\rangle$$

$$(q\neq 0): F = -\frac{1}{4}t^{-2}u + |t|^{q-\frac{3}{2}}G(\omega), \quad \omega = |t|^{\frac{1}{2}-q}u_x.$$

$A_{4.3}$-invariant equations

$$1)\qquad \langle \partial_x, |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u, t\partial_t + x\partial_x + \frac{1}{2}u\partial_u\rangle :$$

$$F = -\frac{1}{4}t^{-2}u + |t|^{-\frac{3}{2}}G(\omega), \quad \omega = |t|^{\frac{1}{2}}u_x;$$

$$2)\qquad \langle \partial_x, t\partial_u, \partial_u, t\partial_t + x\partial_x\rangle : \quad F = t^{-2}G(\omega),\ \omega = tu_x;$$

$$3)\qquad \langle e^{kt}\partial_u, \partial_t + ku\partial_u, \beta\partial_x + te^{kt}\partial_u, e^{-kt}\partial_u\rangle\ (k\neq 0,\ \beta>0):$$

$$F = k^2u + 2k\beta^{-1}xe^{kt} + e^{kt}G(\omega), \quad \omega = e^{-kt}u_x;$$

$$4)\qquad \langle e^{x+kt}\partial_u, e^{\eta}\partial_u, \alpha(\partial_x + u\partial_u) + 2kte^{\eta}\partial_u, -\frac{1}{2k}(\partial_t + k\partial_x)\rangle$$

$$(\alpha\neq 0,\ k>0): \ F = (k^2-1)u - 4k^2\alpha^{-1}\eta e^{\eta} + e^{\eta}G(\omega),$$

$$\omega = e^{-\eta}(u_x - u), \eta = x - kt.$$

$A_{4.4}$-invariant equations

$$1)\qquad \langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u, \partial_x, t\partial_t + x\partial_x + \left[\frac{3}{2}u - x|t|^{\frac{1}{2}}\ln|t|\right]\partial_u\rangle :$$

$$F = \frac{1}{4}t^{-2}u + |t|^{-\frac{1}{2}}G(\omega),\ \omega = |t|^{-\frac{1}{2}}u_x + \frac{1}{2}\ln^2|t|.$$

$A_{4.5}$-invariant equations

$$1)\qquad \langle \partial_x, |t|^{m-\alpha}\partial_u, |t|^{1-m+\alpha}\partial_u, t\partial_t + x\partial_x + mu\partial_u\rangle$$

$$(m\neq \frac{1}{2}(1+\alpha), \frac{1}{2}+\alpha; \alpha\neq 0):$$

$$F = (m-\alpha)(m-\alpha-1)t^{-2}u + |t|^{m-2}G(\omega),\ \omega = |t|^{1-m}u_x.$$

$A_{4.6}$-invariant equations

$$1)\qquad \langle \partial_x, |t|^{\frac{1}{2}}\sin(q^{-1}\ln|t|)\partial_u, |t|^{\frac{1}{2}}\cos(q^{-1}\ln|t|)\partial_u, qt\partial_t + qx\partial_x$$

$$\left(\frac{1}{2}q + p\right)u\partial_u\rangle\ (q\neq 0, p\geq 0):$$

$$F = -\left(\frac{1}{4} + q^{-2}\right)t^{-2}u + |t|^{q^{-1}(p-\frac{3}{2}q)}G(\omega), \quad \omega = |t|^{q^{-1}(\frac{1}{2}q-p)}u_x.$$

$A_{4.7}$-invariant equations

$$1)\qquad \langle \partial_u, -t\partial_u, \partial_t + k\partial_x, t\partial_t + x\partial_x + \left(2u - \frac{1}{2}t^2\right)\partial_u\rangle\ (k\geq 0):$$

$$F = -\ln|\eta| + G(\omega), \quad \omega = \eta^{-1}u_x, \quad \eta = x - kt.$$

$A_{4.8}$-invariant equations

1) $\langle \partial_t + \epsilon u\partial_u, \partial_x, e^x\partial_u, te^x\partial_u\rangle$ $(\epsilon = 0; 1)$:
$F = -u + e^{\epsilon t}G(\omega), \quad \omega = e^{-\epsilon t}(u_x - u);$

2) $\langle |x|^{m-q}\partial_u, \partial_t, t|x|^{m-q}\partial_u, t\partial_t + x\partial_x + mu\partial_u\rangle$ $(q \neq 0,\ m \in \mathbb{R})$:
$F = -(m-q)(m-q-1)x^{-2}u + |x|^{m-2}G(\omega),$
$\omega = |x|^{1-m}[u_x - (m-q)x^{-1}u];$

3) $\langle \partial_t + k\partial_x, \partial_u, t\partial_u, t\partial_t + x\partial_x + qu\partial_u\rangle$ $(k > 0, q \in \mathbb{R})$:
$F = |\eta|^{q-2}G(\omega),\ \omega = |\eta|^{1-q}u_x, \quad \eta = x - kt;$

4) $\langle x^{-1}\partial_u, \partial_t + \partial_x - x^{-1}u\partial_u, tx^{-1}\partial_u, t\partial_t + x\partial_x\rangle$:
$F = 2x^{-1}u_x + x^{-1}(t-x)^{-1}G(\omega), \quad \omega = xu_x + u;$

5) $\langle \partial_u, -t\partial_u, \partial_t + k\partial_x + u\partial_u, \alpha\partial_x + u\partial_u\rangle$ $(\alpha \neq 0, k \geq 0)$:
$F = \exp(\alpha^{-1}\eta + t)G(\omega), \quad \omega = \exp(-\alpha^{-1}\eta - t)u_x,\ \eta = x - kt.$

$A_{4.10}$-invariant equations

1) $\langle \sin t\partial_u, \cos t\partial_u, \partial_x + u\partial_u, \partial_t + k\partial_x\rangle(k \geq 0)$:
$F = -u + e^{\eta}G(\omega), \quad \omega = e^{-\eta}u_x,\ \eta = x - kt.$

In the above formulas $G = G(\omega)$ is an arbitrary function satisfying the condition $F_{u_x u_x} \neq 0$.

7 Symmetry reduction and solutions of nonlinear wave equations

Among various applications of Lie symmetry groups the most prominent and remarkable one is a possibility to construct exact solutions of nonlinear PDEs. The basic idea is reducing multi-dimensional differential equations to ordinary differential equations via special ansatzes (invariant solutions). A regular (but not the only) way to derive those ansatzes is to utilize symmetry group admitted by the equation under study (for more details see, e.g., [14, 15]). Though the obtained ordinary differential equations are, as a rule, nonlinear, they possess in many cases a residual symmetry allowing for constructing their general or particular solutions. Inserting the latter into the corresponding ansatz yields the exact solution of initial nonlinear multi-dimensional PDE. This method is often referred to in the literature as symmetry reduction of PDEs.

We apply the symmetry reduction approach to derive the families of exact solutions of nonlinear wave equations (2.7) having the richest symmetry properties.

To perform reduction of PDEs (2.7) to ordinary differential equations we need to obtain all inequivalent one-dimensional subalgebras of the symmetry algebras of the equations under study. What is more, basis operators of the said one-dimensional algebras

$$\tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u,$$

have to obey the following restriction [14]:

$$|\tau| + |\xi| \neq 0 \tag{7.1}$$

in some open domain Ω of the space $V = \mathbb{R}^2 \times \mathbb{R}^1$ of independent $\mathbb{R}^2 = \langle t, x\rangle$ and dependent $\mathbb{R}^1 = \langle u\rangle$ variables.

As we proved in the previous sections, equations

$$u_{tt} = u_{xx} - u^{-1}u_x^2; \tag{7.2}$$
$$u_{tt} = u_{xx} + e^{u_x}; \tag{7.3}$$
$$u_{tt} = u_{xx} + m\ln|u_x| \ (m>0); \tag{7.4}$$
$$u_{tt} = u_{xx} + |u_x|^k \ (k \neq 0, 1). \tag{7.5}$$

enjoy the highest symmetry properties amongst PDEs of the form (2.7).

The first step of symmetry reduction algorithm is classifying one-dimensional subalgebras of the invariance algebras of the above equations taking into account constraint (7.1). To classify one-dimensional subalgebras we utilize the method suggested in [45] and the lists of one-dimensional subalgebras of four-dimensional subalgebras obtained in [45].

Equation (7.2) admits the algebra

$$A_5^1 = (A_{3.3} \oplus A_1) \subset\!\!\!\!+ \langle e_5 \rangle,$$

where $A_{3.3} = \langle e_1, e_2, e_3\rangle = \langle u\partial_u, \partial_x, xu\partial_u\rangle$, $A_1 = \langle e_4\rangle = \langle \partial_t\rangle$, $e_5 = t\partial_t + x\partial_x$.

In what follows we will need the commutation relations of the basis operators of the algebra $A_{3.3} \oplus A_1$ with the operator e_5:

$$[e_1, e_5] = 0, \quad [e_2, e_5] = e_2, \quad [e_3, e_5] = -e_3, \quad [e_4, e_5] = e_4.$$

According to [45] one-dimensional subalgebras of $A_{3.3} \oplus A_1$ defined within action of inner automorphism group of this algebra read as $\langle e_1\rangle$, $\langle e_1 + \alpha e_4\rangle$, $\langle e_4\rangle$, $\langle e_2\rangle$,$\langle e_2 + \alpha e_4\rangle$, $\langle e_3\rangle$, $\langle e_3 + \alpha e_4\rangle$, $\langle e_2 + \beta e_3\rangle$, $\langle e_2 + \beta e_3 + \alpha e_4\rangle$ $(\alpha, \beta \neq 0)$. The above subalgebras can be further simplified by using transformation group generated by the operator e_5. For example, using the Campbell-Hausdorff formula we transform $e_1 + \alpha e_4$ as follows:

$$\exp(\theta e_5)(e_1 + \alpha e_4)\exp(-\theta e_5) = e_1 + \alpha e^{\theta} e_4.$$

Consequently, putting $\theta = -\ln|\alpha|$ we simplify $e_1 + \alpha e_4$ to become $e_1 \pm e_4$. Similarly, we prove that we can put $\alpha = \pm 1$ in $e_3 + \alpha e_4$ and $\beta = \pm 1$ in $e_2 + \beta e_3$, $e_3 + \beta e_3 + \alpha e_4$.

To complete classification of one-dimensional subalgebras we have to describe all inequivalent subalgebras with non-zero projection on the basis operator e_5, i.e., subalgebras of the form

$$\Lambda = e_5 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4, \quad \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}. \tag{7.6}$$

Utilizing the automorphism $\exp(\theta e_4)$ with properly chosen θ we have $\alpha_4 = 0$ in (7.6). Next, applying transformation $\exp(\theta_1 e_2 + \theta_2 e_3)$ to operator (7.6) reduces it to one of the following operators: $e_5, e_5 + \alpha e_1$ $(\alpha \neq 0)$.

So the list of one-dimensional subalgebras of the five-dimensional algebra A_5^1 determined up to the action of inner automorphism group is exhausted by the following algebras: $\langle e_1\rangle$, $\langle e_1 \pm e_4\rangle$, $\langle e_4\rangle$, $\langle e_2\rangle$, $\langle e_2 + \alpha e_4\rangle$, $\langle e_3\rangle$, $\langle e_3 \pm e_4\rangle$, $\langle e_2 \pm e_3\rangle$, $\langle e_2 \pm e_3 + \alpha e_4\rangle, \langle e_5\rangle, \langle e_5 + \alpha e_1\rangle$ $(\alpha \neq 0)$. By direct verification we prove that the basis operators of the algebras $\langle e_1\rangle, \langle e_3\rangle$ do not satisfy condition (7.1).

Finally, we make use of the fact that the discrete groups of transformations

$$\bar{t} = -t, \quad \bar{x} = x, \quad v = u;$$
$$\bar{t} = t, \quad \bar{x} = -x, \quad v = u;$$
$$\bar{t} = -t, \quad \bar{x} = -x, \quad v = u,$$

also belong to the equivalence group of (7.2). Using the above transformations enables to further simplify the optimal system of inequivalent subalgebras

$$\begin{gathered}\langle e_1 + e_4\rangle, \quad \langle e_4\rangle, \quad \langle e_2\rangle, \quad \langle e_2 + \alpha e_4\rangle, \ \langle e_3 + e_4\rangle, \\ \langle e_2 \pm e_3\rangle, \quad \langle e_2 \pm e_3 + \alpha e_4\rangle, \langle e_5\rangle, \langle e_5 + \alpha e_1\rangle \ (\alpha > 0).\end{gathered} \tag{7.7}$$

The second step of the method of symmetry reduction is constructing the complete set of invariants $f(t,x,u)$ for each inequivalent one-dimensional subalgebra. As a typical example, we consider the case of subalgebra $\langle e_1 + e_4 \rangle$. To construct its invariants we need to integrate the first-order PDE

$$(e_1 + e_4) \circ F(t,x,u) = 0$$

or

$$uF_u + F_t = 0.$$

The complete set of first integrals of the above equation reads as $\omega_1 = x, \quad \omega_2 = e^{-t}u$. Hence we get the general form of invariant solution (ansatz) $\omega_2 = \varphi(\omega_1)$. Solving this equation with respect to u we finally have

$$u = e^t \varphi(x). \tag{7.8}$$

Inserting (7.8) into (7.2) yields ordinary differential equation for unknown function φ

$$\varphi'' - \varphi^{-1}(\varphi')^2 - \varphi = 0,$$

which is easily integrated

$$\varphi = \exp\left[\frac{1}{2}x^2 + C_1 x + C_2\right], \quad C_1, C_2 \in \mathbb{R}.$$

Inserting the so obtained expression for ϕ into ansatz (7.8) yields the explicit form of invariant solution of equation (7.2)

$$u = \exp\left[t + \frac{1}{2}x^2 + C_1 x + C_2\right], \quad C_1, C_2 \in \mathbb{R}.$$

Analysis of the remaining subalgebras from (7.7) yields the following results. Invariant solutions of (7.2) that correspond to the algebras $\langle e_4 \rangle$, $\langle e_2 \rangle$, $\langle e_2 + \alpha e_4 \rangle (\alpha > 0)$ read as

$$\begin{aligned} u &= \exp[C_1 x + C_2], \\ u &= C_1 t + C_2, \\ u &= C_2 |t - \alpha x + C_1|^{1-\alpha^2}, \end{aligned}$$

where $C_1, C_2 \in \mathbb{R}$.

The ansatz corresponding to the algebra $\langle e_3 + e_4 \rangle$ has the form

$$u = e^{tx}\varphi(x).$$

It reduces (7.2) to ordinary differential equation

$$\varphi'' - \varphi^{-1}(\varphi')^2 - x^2\varphi = 0,$$

whose general solution has the form

$$\varphi = \exp\left[\frac{1}{12}x^4 + C_1 x + C_2\right], \quad C_1, C_2 \in \mathbb{R}.$$

Inserting the above expression into the corresponding ansatz we get another family of exact solutions of nonlinear wave equation (7.2)

$$u = \exp\left[tx + \frac{1}{12}x^4 + C_1 x + C_2\right], \quad C_1, C_2 \in \mathbb{R}.$$

A similar analysis applied to the algebra $\langle e_2 + \epsilon e_3 \rangle$ ($\epsilon = \pm 1$) yields two families of exact solutions (7.2):

$$\begin{aligned} u &= C_1 \exp\left(-\frac{1}{2}x^2\right)\cos(t + C_2), \quad \epsilon = -1; \\ u &= C_1 \exp\left(\frac{1}{2}x^2\right)\cosh(t + C_2), \quad \epsilon = 1; \end{aligned}$$

where $C_1, C_2 \in \mathbb{R}$.

The ansatz invariant under the algebra $\langle e_2 + \epsilon e_3 + \alpha e_4 \rangle$ ($\epsilon = \pm 1; \alpha > 0$) reads as

$$u = \exp\left(\frac{1}{2}\epsilon x^2\right)\varphi(\eta), \quad \eta = t - \alpha x.$$

It reduces PDE (7.2) to equation

$$(\alpha^2 - 1)\varphi'' - \alpha^2\varphi^{-1}(\varphi')^2 + \epsilon\varphi = 0,$$

whose general solution is given by one of formulas below

$$\begin{aligned} u &= \exp(C_1 \pm \eta), \quad \alpha = 1, \epsilon = 1; \\ u &= C_2\left[\cos\left(C_1 + \frac{\eta}{\alpha^2 - 1}\right)\right]^{1-\alpha^2}, \quad \alpha > 0,\ \alpha \neq 1, \quad \epsilon = -1; \\ u &= C_2\left[\cosh\left(C_1 + \frac{\eta}{1 - \alpha^2}\right)\right]^{1-\alpha^2}, \quad \alpha > 0,\ \alpha \neq 1, \quad \epsilon = 1; \end{aligned}$$

where $C_1, C_2 \in \mathbb{R}$. The corresponding invariant solutions of (7.2) have the form

$$\begin{aligned} u &= \exp\left[C_1 + \frac{1}{2}x \pm (t - x)\right]; \\ u &= C_2 \exp\left(-\frac{1}{2}x^2\right)\left[\cos\left(C_1 + \frac{t - \alpha x}{\alpha^2 - 1}\right)\right]^{1-\alpha^2}, \quad \alpha > 0,\ \alpha \neq 1; \\ u &= C_2 \exp\left(-\frac{1}{2}x^2\right)\left[\cosh\left(C_1 + \frac{t - \alpha x}{1 - \alpha^2}\right)\right]^{1-\alpha^2}, \quad \alpha > 0,\ \alpha \neq 1; \end{aligned}$$

where $C_1, C_2 \in \mathbb{R}$.

The algebras $\langle e_5 \rangle, \langle e_5 + \alpha e_1 \rangle$ ($\alpha > 0$) give rise to the so called auto-model ([14]) solutions of (7.2). Solution invariant under $\langle e_5 \rangle$ reads as

$$\begin{aligned} u &= \exp\left\{\int^{\xi}\left(C_1(\eta^2 - 1) + \frac{1}{4}(\eta^2 - 1)\ln\left|\frac{1+\eta}{1-\eta}\right| - \frac{1}{2}\eta\right]^{-1} d\eta + C_2\right\}, \\ & \xi = tx^{-1}, \quad C_1, C_2 \in \mathbb{R}. \end{aligned}$$

Solution invariant under the algebra $\langle e_5 + \alpha e_1 \rangle$ ($\alpha > 0$) has the form

$$u = |t|^{\alpha}\varphi(\xi), \quad \xi = tx^{-1}.$$

Inserting this expression into (7.2) yields ordinary differential equation

$$\xi^2(\xi^2 - 1)\varphi'' - \xi^4\varphi^{-1}(\varphi')^2 + 2\xi(\xi^2 - \alpha)\varphi' - \alpha(\alpha - 1)\varphi = 0.$$

If $\alpha = 1$, then the general solution of the above equation is of the form

$$\begin{aligned}\varphi &= \exp\left\{\int^{\xi}\left[C_1\eta^2 + \frac{1}{2}\eta\ln\left|\frac{1+\eta}{1-\eta}\right| - \eta\right]^{-1} d\eta + C_1\right\},\\ &\xi = tx^{-1}, \quad C_1, C_2 \in \mathbb{R}.\end{aligned}$$

Provided $\alpha > 0, \alpha \neq 1$, the change of variables

$$\varphi = \exp f, \quad f = f(\xi)$$

reduces the equation for ϕ to become

$$\xi^2(\xi^2 - 1)f'' - \xi^2(f')^2 + 2\xi(\xi^2 - \alpha)f' - \alpha(\alpha - 1) = 0.$$

Now making the change of variables, $f' = g, g = g(\xi)$ we arrive at the Riccatti equation

$$\xi^2(\xi^2 - 1)g' = \xi^2 g^2 - 2\xi(\xi^2 - \alpha)g + \alpha(\alpha - 1). \tag{7.9}$$

In view of the above we have the two families of exact solutions of nonlinear wave equation (7.2):

$$\begin{aligned}u &= t\exp\left\{\int^{\xi}\left[C_1\eta^2 + \frac{1}{2}\eta\ln\left|\frac{1+\eta}{1-\eta}\right| - \eta\right]^{-1} d\eta + C_2\right\},\\ &\xi = tx^{-1}, \quad C_1, C_2 \in \mathbb{R};\\ u &= |t|^{\alpha}\exp\left[\int^{\xi} g(\eta)d\eta + C_1\right],\end{aligned}$$

where $\alpha > 0, \ \alpha \neq 1, \quad \xi = tx^{-1}, C_1 \in \mathbb{R}$, and the function $g = g(\xi)$ is a solution of (7.9).

Now we turn to nonlinear wave equation (7.3). Its maximal symmetry algebra is $A_5^2 = (A_{3.3} \oplus A_1) \supset\!\!\!\!+ \langle e_5\rangle$, where $A_{3.3} = \langle e_1, e_2, e_3\rangle = \langle \partial_u, \partial_t, t\partial_u\rangle$, $A_1 = \langle e_4\rangle = \langle \partial_x\rangle$, $e_5 = t\partial_t + x\partial_x + (u - x)\partial_u$. What is more, the basis operators of the algebra $A_{3.3} \oplus A_1$ obey the following commutation relations with e_5:

$$[e_1, e_5] = e_1, \quad [e_2, e_5] = e_2, \quad [e_3, e_5] = 0, \quad [e_4, e_5] = e_4 - e_1.$$

Using these and taking into account condition (7.1) we obtain the following system of optimal one-dimensional subalgebras of the algebra A_5^2:

$$\begin{aligned}&\langle e_4\rangle, \ \langle e_2\rangle, \ \langle e_2 + \alpha e_4\rangle, \ \langle e_3 + \alpha e_4\rangle, \langle e_2 \pm e_3\rangle,\\ &\langle e_2 + \beta e_3 + \alpha e_4\rangle, \ \langle e_5\rangle, \ \langle e_5 + \alpha e_3\rangle \ (\alpha > 0, \beta \neq 0).\end{aligned}$$

Skipping the intermediate computations we give the final result, the families of exact solutions of nonlinear equation (7.3):

$$\begin{aligned}&\langle e_4\rangle : u = C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R};\\ &\langle e_2\rangle : u = (x + C_1)[1 - \ln|x + C_1|] + C_2, \quad C_1, C_2 \in \mathbb{R};\\ &\langle e_2 + \alpha e_4\rangle : u = (x - \alpha t)\ln|1 - \alpha^2|\\ &-(x - \alpha t + C_1)[\ln|x - \alpha t + C_1| - 1] + C_2,\\ &\quad \alpha > 0, \alpha \neq 1, \quad C_1, C_2 \in \mathbb{R};\\ &\langle e_3 + \alpha e_4\rangle : u = \alpha^{-1}tx + \alpha^2\exp(\alpha^{-1}t) + C_1 t + C_2, \quad \alpha > 0, C_1, C_2 \in \mathbb{R};\end{aligned}$$

$$
\begin{aligned}
&\langle e_2+\epsilon e_3\rangle : u=\frac{1}{2}\epsilon t^2+\varphi(x), \quad \epsilon=\pm 1,\ \varphi'=y(x),\\
&y-\ln|\epsilon-e^y|=\epsilon x+C_1, \quad C_1\in\mathbb{R};\\
&\langle e_2+\beta e_3+e_4\rangle : u=\frac{1}{2}\beta t^2+(x-t)\ln|\beta|+C_1, \quad \beta\neq 0, \quad C_1\in\mathbb{R};\\
&\langle e_2+\beta e_3+\alpha e_4\rangle : u=\frac{1}{2}\beta t^2+\varphi(\eta), \quad \eta=x-\alpha t, \quad \varphi=y'(\eta),\\
&y-\ln|\beta-e^y|=\beta(1-\alpha^2)^{-1}\eta+C_1, \quad \alpha>0, \alpha\neq 1, \beta\neq 0, C_1\in\mathbb{R};\\
&\langle e_5\rangle : u=-x\ln|x|+C_1t+x, \quad C_1\in\mathbb{R};\\
&\langle e_5+\alpha e_3\rangle : u=(\alpha t-x)\ln|x|+x\varphi(\xi), \quad \xi=tx^{-1},\ \alpha>0,\\
&(\xi^2-1)\varphi''+\exp[-\xi\varphi'+\varphi+\alpha\xi-1]=1+\alpha\xi.
\end{aligned}
$$

Equation (7.4) admits the algebra $A_5^3=(A_{3.5}\oplus A_1)\ni\langle e_5\rangle$, and $A_{3.5}=\langle e_1,e_2,e_3\rangle=\langle\partial_u,\partial_t,t\partial_u\rangle$, $A_1=\langle e_4\rangle=\langle\partial_x\rangle$, $e_5=t\partial_t+x\partial_x+\left(2u+\frac{1}{2}mt^2\right)\partial_u$. Its optimal system of one-dimensional subalgebras has the form

$$
\langle e_1+e_4\rangle,\ \langle e_4\rangle,\ \langle e_2\rangle,\ \langle e_2+\alpha e_4\rangle, \langle e_3+\alpha e_4\rangle, \langle e_5\rangle\ (\alpha>0).
$$

Below we give the list of exact solutions of (7.4) invariant under the above subalgebras.

$$
\begin{aligned}
&\langle e_1+e_4\rangle : u=x+C_1t+C_2, C_1, C_2\in\mathbb{R};\\
&\langle e_4\rangle : u=C_1t+C_2, C_1, C_2\in\mathbb{R};\\
&\langle e_2\rangle : u=\varphi(x), \quad \varphi'=f(x), \ \int\frac{df}{\ln|f|}=-mx+C_1,\ C_1\in\mathbb{R};\\
&\langle e_2+\alpha e_4\rangle : u=x-t+C_1, C_1\in\mathbb{R},\ \text{if}\ \alpha=1;\\
&u=\varphi(\eta),\ \eta=x-\alpha t,\ \varphi'=f(\eta),\\
&\int\frac{df}{\ln|f|}=-\frac{m}{1-\alpha^2}(x-\alpha t)+C_1,\ C_1\in\mathbb{R},\ \text{if}\ \alpha>0, \alpha\neq 1;\\
&\langle e_3+\alpha e_4\rangle : u=\alpha^{-1}tx+\frac{1}{2}mt^2(1+\ln\alpha)+\frac{1}{2}mt^2\left(\ln|t|-\frac{1}{2}\right)\\
&+C_1t+C_2, \alpha>0, C_1, C_2\in\mathbb{R}.\\
&\langle e_5\rangle : u=\frac{1}{2}m^2\ln|t|+t^2\varphi,\ \text{where}\ \varphi=\varphi(\xi)\ (\xi=tx^{-1})\ \text{satisfy equation}\\
&\xi^2(\xi^2-1)\varphi''+2\xi(\xi^2-2)\varphi'-2\varphi+m\ln|\xi^2\varphi'|-\frac{3}{2}m=0.
\end{aligned}
$$

Note that the particular solution of reduced equation $\varphi=-\xi^{-1}\exp(\frac{3}{2})$ gives rise to the following invariant solution of (7.4):

$$
u=\frac{1}{2}mt^2\ln|t|-tx\exp(\frac{3}{2}).
$$

Equation (7.5) is invariant under the algebra $A_5^4=(A_{3.5}\oplus A_1)\ni\langle e_5\rangle$, and $A_{3.5}=\langle e_1,e_2,e_3\rangle=\langle\partial_u,\partial_t,t\partial_u\rangle$, $A_1=\langle e_4\rangle=\langle\partial_x\rangle$, $e_5=t\partial_t+x\partial_x+\frac{k-2}{k-1}u\partial_u$, $k\neq 0;1$. The optimal system of one-dimensional subalgebras of this algebra reads as $\langle e_5\rangle, \langle e_1+e_4\rangle$, $\langle e_4\rangle, \langle e_2\rangle, \langle e_2+\alpha e_4\rangle$, $\langle e_3+e_4\rangle$,$\langle e_2\pm e_3\rangle$, $\langle e_2\pm e_3+\alpha e_4\rangle$ $(\alpha>0)$. If $k=2$, then the above list should include also the algebra $\langle e_5+\alpha e_1\rangle$ $(\alpha>0)$.

Below we give exact solutions of nonlinear wave equation (7.5) invariant under the above algebras.

$$\langle e_1+e_4\rangle : u = \frac{1}{2}t^2 + x + C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R};$$

$$\langle e_4\rangle : u = C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R};$$

$$\langle e_2\rangle : u = (2-k)^{-1}(C_1 + (k-1)x)^{\frac{2-k}{1-k}} + C_2, C_1, C_2 \in \mathbb{R},$$

if $k \neq 2$;

$$u = C_2 \ln|x - C_1|, \ C_1, C_2 \in \mathbb{R}, \text{ if } k = 2;$$

$$\langle e_2 + \alpha e_4\rangle : u = C, \quad C \in \mathbb{R}, \text{ if } \alpha = 1;$$

$$u = (1-\alpha^2)\ln|C_1 - x + \alpha t| + C_2, \ C_1, C_2 \in \mathbb{R},$$

if $k = 2, \alpha > 0, \alpha \neq 1$;

$$u = \frac{1-k}{2-k}\left(\frac{1-k}{\alpha^2-1}\right)^{\frac{1}{1-k}} |x - \alpha t + C_1|^{\frac{2-k}{1-k}} + C_2, \ C_1, C_2 \in \mathbb{R},$$

if $k \neq 0, 1, 2; \alpha > 0, \alpha \neq 1$;

$$\langle e_3 + e_4\rangle : u = tx + t(\ln|t| - 1) + C_1 t + C_2, \ C_1, C_2 \in \mathbb{R},$$

if $k = -1$;

$$u = tx - \ln|t| + C_1 t + C_2, \ C_1, C_2 \in \mathbb{R}, \text{ if } k = -2;$$

$$u = tx + (k^2 + 3k + 2)^{-1}|t|^{k+2} + C_1 t + C_2, \ C_1, C_2 \in \mathbb{R},$$

if $k \neq 0, 1, -1, -2$;

$$\langle e_2 + \epsilon e_3\rangle : u = \frac{1}{2}\epsilon t^2 \varphi, \ \varphi = \varphi(x) = \int^x f(\eta)d\eta + C_2,$$

where f is defined by

$$\int \frac{df}{\epsilon - |f|^k} = \eta + C_1, C_1, C_2 \in \mathbb{R}, \ \epsilon = \pm 1;$$

$$\langle e_2 + \epsilon e_3 + \alpha e_4\rangle : u = \frac{1}{2}\epsilon t^2 + \varphi, \ \varphi = \varphi(\eta) = \int^\eta f(z)dz + C_2,$$

$\eta = x - \alpha t$, where f is defined by

$$\int \frac{df}{\epsilon - |f|^k} = (1-\alpha^2)^{-1} z + C_1, C_1, C_2 \in \mathbb{R}, \ \epsilon = \pm 1, \ \alpha > 0;$$

$\langle e_5\rangle : u = |t|^{\frac{k-2}{k-1}}\varphi(\xi), \ \xi = tx^{-1}$, where φ is defined by

$$\xi^2(\xi^2 - 1)\varphi'' + 2\xi\left(\xi^2 - \frac{k-2}{k-1}\right)\varphi' + (-1)^k \xi^{2k}|\varphi'|^k +$$

$$+ \frac{k-2}{(k-1)^2}\varphi = 0, \quad \text{if } k \neq 0, 1, 2;$$

$$u = \int^\xi \left[C_1(1-\eta^2) + \frac{1}{4}(1-\eta^2)\ln\left|\frac{1+\eta}{1-\eta}\right| - \frac{1}{2}\eta\right]^{-1} d\eta + C_2,$$

$C_1, C_2 \in \mathbb{R}, \xi = tx^{-1}$, if $k = 2$;

$$\langle e_5 + \alpha e_1\rangle : u = \alpha \ln|t| + \varphi, \ \varphi = \varphi(\xi) = \int^\xi f(\eta)d\eta + C,$$

$\xi = tx^{-1}, C \in \mathbb{R}$, $f = f(\eta)$ is a solution of Riccatti equation

$\eta^2(\eta^2 - 1)f' + \eta^4 f^2 + 2\eta^3 f + \alpha = 0, \alpha > 0$, if $k = 2$.

8 Concluding remarks

Let us briefly summarize the results obtained in this paper.

We prove that the problem of group classification of the general quasi-linear hyperbolic type equation (1.1) reduces to classifying equations of more specific forms

$$
\begin{array}{ll}
\text{I.} & u_{tt} = u_{xx} + F(t,x,u,u_x), \quad F_{u_x u_x} \neq 0; \\
\text{II.} & u_{tt} = u_{xx} + g(t,x,u)u_x + f(t,x,u), \quad g_u \neq 0; \\
\text{III.} & u_{tx} = g(t,x)u_x + f(t,x,u), \ g_x \neq 0, \ f_{uu} \neq 0; \\
\text{IV.} & u_{tx} = f(t,x,u), \quad f_{uu} \neq 0.
\end{array}
$$

The cases of PDEs that are essentially nonlinear in u_x (the class of PDEs I) and either linear in u_x or do not depend on u_x (the classes II - IV) need to be considered separately.

If we denote as $\mathcal{DE}$ the set of PDEs II – III, then the results of application of our algorithm for group classification of equations I – IV can be summarized as follows.

1) We perform complete group classification of the class $\mathcal{DE}$. We prove that the Liouville equation has the highest symmetry properties among equations from $\mathcal{DE}$. Next, we prove that the only equation belonging to this class and admitting the four-dimensional invariance algebra is the nonlinear d'Alembert equations. It is established that there are twelve inequivalent equations from $\mathcal{DE}$ invariant under three-dimensional Lie algebras. We give the lists of all inequivalent equations from $\mathcal{DE}$ that admit one- and two-dimensional symmetry algebras.

2) We have studied the structure of invariance algebras admitted by nonlinear equations from the class I. It is proved, in particular, that their invariance algebras are necessarily solvable.

3) We perform complete group classification of nonlinear equations from the class of PDEs I. We prove that the highest symmetry algebras admitted by those equations are five-dimensional and construct all inequivalent classes of equations invariant with respect to five-dimensional Lie algebras. We also construct all inequivalent equations of the form I admitting one-, two-, three- and four-dimensional Lie algebras.

The results of group classification of the class of nonlinear wave equations (1.1) are utilized for constructing their explicit solutions. Namely, we perform symmetry reduction of all equations (1.1) admitting five-dimensional symmetry algebras to ordinary differential equations and constructed multi-parameter families of their exact solutions.

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